Upper powerdomain of quasicontinuous domains

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Joint work with Yuxu Chen and Hui Kou

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Some history

1976 **Plotkin** developed a powerdomain construction (aka convex powerdomain). The desire for such a construction arises when considering programming languages with nondeterministic features or parallel features treated in a nondeterministic way.

1978 **Smyth** introduced the Smyth (aka upper) powerdomain which was useful in modelling demonic nondeterminism.

1979 **Hennessy** & **Plotkin** suggested that powerdomains were free algebras with respect to certain (in-)equational theories in the realm of domain theory.

1991 **Heckmann** introduced a novel upper power domain construction by means of strongly compact sets and obtained its commutation with the lower construction.

1993 **Schalk** proposed a new representation of Smyth powerdomain as the set of Scott open filters over a dcpo, but this representation did not work for any dcpo.

2008 **Jung, Moshier & Vickers** constructed the general free dcpo-algebras over dcpos through the dcpo presentation by an algebraic method.

A problem

- However, there is no further study on the topological representation of the upper powerdomains of general dcpos.
- (Refer to [Abramsky-Jung 1994]) For any continuous dcpo D, the set of all nonempty Scott compact saturated subsets of L with reversed inclusion order ($\mathcal{Q}(L)$) is the upper powerdomain over L.
- But it is unknown that what is the topological representation of the upper (Smyth) powerdomain over a non-continuous dcpo. Specifically, when does the upper powerdomain over a quasicontinuous dcpo L coincides with $\mathcal{Q}(L)$?

In this talk, we aim to deal with this problem.

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Continuous dcpo

Definition

Let D be a partially ordered set (poset). If any directed subset of D has a supremum, then we call D a directed complete poset (dcpo).

A dcpo D is called continuous if the way-below relation on D is approximating.

- For any $x, y \in D$, x is way-below y (denoted by $x \ll y$) if for any directed subset A of D with $y \leq \sup^{\uparrow} A$, then there exists some $a \in A$ such that $x \leq a$.
- the way-below relation is said to be approximating if for any $x \in D$, $\{d \in D : d \ll x\}$ is directed and $\sup\{d \in D : d \ll x\} = x$.

A function f between two dcpos D and E is called **Scott continuous** if it is monotone and preserves all directed supremums.

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Scott topology

Definition

Let D be a dcpo. A subset U of D is said to be Scott open if

- U is upward closed, i.e., $U = \uparrow U$;
- $\sup^{\uparrow} A \in U$ implies $A \cap U \neq \emptyset$ for all directed subsets A of D.

The collection of all Scott open sets on D is called the Scott topology of D and it is denoted by $\sigma(D)$. Usually we use ΣD to denote the dcpo D with its Scott topology.

The Scott topology can be generalized to the setting of posets, we can replace "directed subsets" by "directed subsets with supremums" in the second condition.

Proposition

Let D and E be dcpos. Then a map $f\colon D \to E$ is Scott continuous iff it is continuous respect to the Scott topology.

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Upper powerdomain

Definition

Let X be a dcpo and $\mathrm{DCPO}(\Sigma, \mathcal{E})$ be the category of dcpo-algebras with signature Σ satisfying $\mathcal{E}$. A pair (A, j) is the free algebra over X with respect to $\mathrm{DCPO}(\Sigma, \mathcal{E})$ if A is an object of $\mathrm{DCPO}(\Sigma, \mathcal{E})$, $j\colon X \to A$ is Scott continuous, and any Scott continuous map $f\colon X \to B$, an object of $\mathrm{DCPO}(\Sigma, \mathcal{E})$, extends uniquely to a dcpo-morphism $h\colon A \to B$ such that $h \circ j = f$.

Let $\Sigma_0 = \{+\}$, and $\mathcal{E}_0$ consists of the following:

(1) Commutativity: $x + y = y + x$;

(2) Associativity: $(x + y) + z = x + (y + z)$;

(3) Idempotence: $x + x = x$;

(4) Deflation: $x + y \leq x$.

The objects of $\mathrm{DCPO}(+, \mathcal{E}_0)$ are called deflationary semilattices. The free deflationary semilattice over a dcpo L is called the **upper powerdomain** of L, denoted by $P_U(L)$.

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The lower powerdomain

- If we replace the fourth condition in $\mathcal{E}_0$ by $x + y \geq x$ (inflation), we obtain the concept of lower powerdomain. The free inflationary semilattice over a dcpo L is called the **lower powerdomain** of L, denoted by $P_L(L)$.
- It turns out that $P_L(L)$ has a very direct topological representation. For example, see [Abramsky-Jung 1994].

Theorem

Let L be a dcpo and $\Gamma(L)$ be the set of nonempty Scott closed sets of L with inclusion order $(+ = \cup)$. Then $\Gamma(L)$ with the map $\eta = \lambda x. \downarrow x \colon L \to \Gamma(L)$ is the free inflationary semilattice.

But the case of upper powerdomains is more intricate.

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But the case of upper powerdomains is more intricate.

Continuous dcpo setting

Theorem ([Abramsky-Jung 1994])

Let L be any continuous dcpo and $\mathcal{Q}(L)$ denote the set of nonempty Scott compact saturated subsets of L with reversed inclusion order $(+ = \cup)$. Then $\mathcal{Q}(L)$ with the map $\eta = \lambda x.{\uparrow}x \colon L \to \mathcal{Q}(L)$ is the free deflationary semilattice.

In order to deal with the case of non-continuous dcpos, we need tools from directed spaces and Scott-completions.

The strategy is that **we work in a bigger environment—directed spaces, and then we use Scott-completions to obtain a desired dcpo**.

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Directed space

Directed spaces are introduced by [Kou-Yu 2015] to extend Scott spaces of dcpos to general topological spaces. [Erné 2009] introduced the concept of monotone determined spaces. Directed spaces are equivalent to **monotone determined** T_0 spaces.

Definition (specialization order)

Let X be a T_0 space. There is a natural ordering (**specialization order**) $\leq$ on X, $x \leq y$ iff x is included in the closure of $\{y\}$.

Definition (directed open)

Let X be a T_0 space. For every directed set D of X, D can be naturally viewed as a monotone net $(d)_{d\in D}$ with respect to the specialization order.
We say that U is a **directed open** set if for any $x \in U$ and any directed set D of X such that the net $(d)_{d\in D}$ converges to x in X, D intersects U.

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Some examples

Example

- Any poset with Scott topology. In particular, every Scott space (a dcpo with its Scott topology) is a directed space.
- Any locally finitary compact space. Hence every C-space is a directed space. In particular every Alexandroff space is a directed space. Notice that the topology on a directed space may not be the Scott topology.
- A non-directed space. The set of natural numbers with cofinite topology.
- Actually, for any T_0 space X, denote $d(X)$ the set of all directed open subsets of X, then $\mathcal{D}(X) = (X, d(X))$ is a directed space. We say that $\mathcal{D}(X)$ is the directed space generated by X.

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Scott completion

Definition ([Zhang-Shi-Li 2021])

A Scott completion (Y, f) of a space X is a Scott space Y together with a continuous map $f\colon X \to Y$ such that for any Scott space Z and continuous map $g\colon X \to Z$, there exists a unique continuous map $\tilde{g}$ satisfying $g = \tilde{g} \circ f$.

Every directed space has a Scott completion.

Theorem ([Zhang-Shi-Li 2021])

Let X be a directed space. $(\tilde{X}, i)$ is a Scott completion of X, which we call the standard Scott completion.

Let X be a directed space, the Scott completion of X is just the D-completion of X because D-completion of X is a Scott space.

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Not every T_0 space has a Scott completion

Example

Let $\mathbb{N}$ be the set of natural numbers endowed with co-finite topology.

Sketch of the proof. Assume that $\mathbb{N}$ has a Scott completion $(\bar{\mathbb{N}}, f)$.

(1) f is injective.
For any $n \neq m$, consider $\mathbb{S} = \{\perp, \top\}(\perp < \top)$ with Scott topology. Define a continuous map $g\colon \mathbb{N} \to \mathbb{S}$ as follows: $g(n) = \perp, g(x) = \top$ for any $x \neq n$. There is a continuous map $\tilde{g}\colon \bar{\mathbb{N}} \to \mathbb{S}$ such that $\tilde{g} \circ f = g$.

(2) For any $n \neq m$ of $\mathbb{N}$, $f(n)$ and $f(m)$ are incomparable.

(3) For any $x \in \bar{\mathbb{N}}, \downarrow x \cap f(\mathbb{N}) \neq \emptyset$.

We conclude that $f(\mathbb{N})$ is the set of minimal elements of $\bar{\mathbb{N}}$. Notice that every subset A of $f(\mathbb{N})$ is Scott closed in $\bar{\mathbb{N}}$. Hence $f(\mathbb{N})$ with its subspace topology is a discrete space.Then $\mathbb{N}$ is a discrete space as well, a contradiction.

Cartesian closedness

Theorem ([Kou-Yu 2015])

DTop *is cartesian closed.*

For any two directed spaces X and Y, the ***categorical product*** *$X \otimes Y$ is isomorphic to $\mathcal{D}(X \times Y)$ where $X \times Y$ is the space of the product topology of X and Y.*

For any two directed spaces X and Y, the ***exponential object*** *Y^X is isomorphic to $\mathcal{D}([X \to Y]_p)$ where $[X \to Y]_p$ is the set of all continuous functions from X to Y with the pointwise topology.*

If we view **DCPO** as a subcategory of **DTop**, the embedding functor preserves the product and exponential object.

Directed continuous map

Theorem ([Kou-Yu 2015])

Let X and Y be two directed spaces. Let Z be a directed space. Then $f: X \otimes Y \longrightarrow Z$ is continuous if and only if it is continuous at each argument separately.

Definition ([Kou-Yu 2015])

Let X, Y be two T_0 spaces. A map $f: X \longrightarrow Y$ is called directed continuous if it is monotone and preserves all limits of directed subsets of X, that is, $(D, x) \in D(X) \Rightarrow (f(D), f(x)) \in D(Y)$.

- It is easy to verify that f is continuous if and only if it is directed continuous.

The directed deflationary semilattice

Definition

Let X be a directed space.

1. A binary operation $\oplus \colon X \otimes X \to X$ is called a deflationary operation if it is continuous and satisfies the following four conditions: $\forall x, y, z$
   (a) $x \oplus y = y \oplus x$;
   (b) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
   (c) $x \oplus x = x$;
   (d) $x \oplus y \leq x$.
2. If $\oplus$ is a deflationary operation on X, then $(X, \oplus)$ is called a directed deflationary semilattice.

Deflationary homomorphism and directed upper powerspace

Definition

Suppose that $(X, \oplus)$ and $(Y, \uplus)$ are two directed deflationary semilattices. A map $f\colon (X, \oplus) \to (Y, \uplus)$ is called a deflationary homomorphism between X and Y, if f is continuous and $f(x \oplus y) = f(x) \uplus f(y)$ holds for all $x, y \in X$.

Definition

Let X be a directed space. A directed space Z is called the directed upper powerspace of X if and only if the following two conditions are satisfied:

- Z is a directed deflationary semilattice, i.e., the meet operation $\wedge$ on Z exists and is continuous;
- There is a continuous map $i\colon X \to Z$ satisfying: for any directed deflationary semilattice $(Y, \wedge)$ and any continuous map $f\colon X \to Y$, there exists a unique deflationary homomorphism $\overline{f}\colon (Z, \wedge) \to (Y, \wedge)$ such that $f = \overline{f} \circ i$.

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A topology on the set of finitely generated subsets

Let X be a directed space. Set $UX = \{\uparrow F : F \subseteq_f X, F = \min(\uparrow F)\}$, which is called the set of all finitely generated upper subsets of X. Define an order $\leq_U$ on UX as follows: $\uparrow F_1 \leq_U \uparrow F_2$ iff $\uparrow F_2 \subseteq \uparrow F_1$.

Definition ([Xie-Kou 2020])

(A convergence on UX) Let X be a directed space and $\mathcal{F}$ be a directed set of UX with respect to $\leq_U$, and any $\uparrow F = \uparrow\{x_1, \cdots, x_n\} \in UX$, we say that $\mathcal{F}$ converges to x if there exist finite directed sets $D_1, \cdots, D_k$ of X such that the following three conditions are satisfied:

(1) For each $x_i \in F$, there exists an $i' \in [1, k]$ such that $D_{i'}$ converges to x_i in X;

(2) For each D_j , there exists an $x_{j_0} \in F$ such that D_j converges to x_{j_0} in X;

(3) $\forall (d_1, \cdots, d_k) \in \prod_{j=1}^{k} D_j$, there is some $\uparrow F' \in \mathcal{F}$ such that $\uparrow F' \subseteq \bigcup_{j=1}^{k} \uparrow d_j$.

By using this convergence, we can generate a topology τ on UX.

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By using this convergence, we can generate a topology τ on UX.

Upper powerdomain over any dcpo

There is a natural binary operation $\cup$ on (UX, τ) such that $(UX, \cup, \tau)$ is a directed deflationary semilattice.

Theorem ([Xie-Kou 2020])

Let X be a directed space. Then $(UX, \cup, \tau)$ with the map $\eta = \lambda x.{\uparrow}x \colon X \to UX$ is the directed upper powerspace over X.

For any directed space X, denote by $\mathcal{U}X = (UX, \tau)$ the directed upper powerspace of X. For any dcpo L, denote by $\mathcal{U}L = \mathcal{U}(\Sigma L)$ for convenience. Recall that the upper powerdomain of L is denoted by $P_U(L)$, and the Scott completion of $\mathcal{U}L$ is denoted by $\tilde{\mathcal{U}}L$.

Theorem

Let L be a dcpo. Then $\tilde{\mathcal{U}}L$ is equal to $P_U(L)$ up to isomorphism.

Upper powerdomain over any dcpo

There is a natural binary operation $\cup$ on (UX, τ) such that $(UX, \cup, \tau)$ is a directed deflationary semilattice.

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Quasicontinuous domains

Definition

A dcpo L is called a **quasicontinuous domain** if $\mathrm{fin}(x) = \{F : F \subseteq_f L, F \ll x\}$ is a directed set with respect to Smyth order and

$$\uparrow x = \bigcap_{F \in \mathrm{fin}(x)} \uparrow F.$$

We have a equivalent description of quasicontinuousness in terms of Scott topology.

Proposition

Let L be a dcpo. Then L is quasicontinuous iff ΣL is locally finitary compact, i.e., for any $a \in L$ and any Scott open subset U containing x, there exists a finite subset $F \subseteq L$ such that

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When does $P_U(L) \cong \mathcal{Q}(L)$?

Theorem

Let L be a quasicontinuous dcpo. The following conditions are equivalent:

(1) $\mathcal{U}L (= (UL, \tau))$ *is a C-space.*

(2) *The topology τ on $\mathcal{U}L$ is equal to the Scott topology on $(UL, \leq_U)$.*

(3) $P_U(L) \cong \mathcal{Q}(L)$.

(4) $P_U(L)$ *is a continuous dcpo.*

(5) $P_U(L)$ *is a quasicontinuous dcpo.*

The following example tells us that there exists some quasicontinuous dcpo L such that the above conditions are all false.

Example

X consists of all finite sequences of $\{0, 1\}$ and an element $\top$. The sequences are ordered by the prefix order, and $\top$ is greater than any finite sequence. But $\tau \neq \sigma(UL, \leq_U)$.

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Let L be a dcpo. We say that L is **strongly quasicontinuous** if for each $x \in L$, there exists finite directed subsets $D_1, \cdots, D_n$ of L such that

- $\forall i \in [1, n], x \leq \sup D_i$.
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Every continuous dcpo is strongly quasicontinuous.
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Corollary

If L is a strongly quasicontinuous dcpo, then $P_U(L) \cong \mathcal{Q}(L)$.

A natural question: **Is the converse result true under the assumption that L is quasicontinuous**?

It seems like that it is not easy to find a quasicontinuous dcpo but not strongly quasicontinuous.

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A little help from a Heckmann's result

Lower and upper powerdomain constructions commute on all dcpos.

Theorem ([Heckmann 1991])

For any dcpo L, $P_U(P_L(L)) \cong P_L(P_U(L))$.

Recall that $P_L(L)$ is just the set of nonempty Scott closed subsets of D.

Proposition

Let L be a quasicontinuous dcpo such that $P_U(L) \cong \mathcal{Q}(L)$. Then $P_L(L)$ is also quasicontinuous and $P_U(P_L(L)) \cong \mathcal{Q}(P_L(L))$.

Proof.

It is well-known that $P_L(L)$ is quasicontinuous. $P_U(P_L(L)) \cong P_L(P_U(L)) \cong P_L(\mathcal{Q}(L))$ is continuous. $\square$

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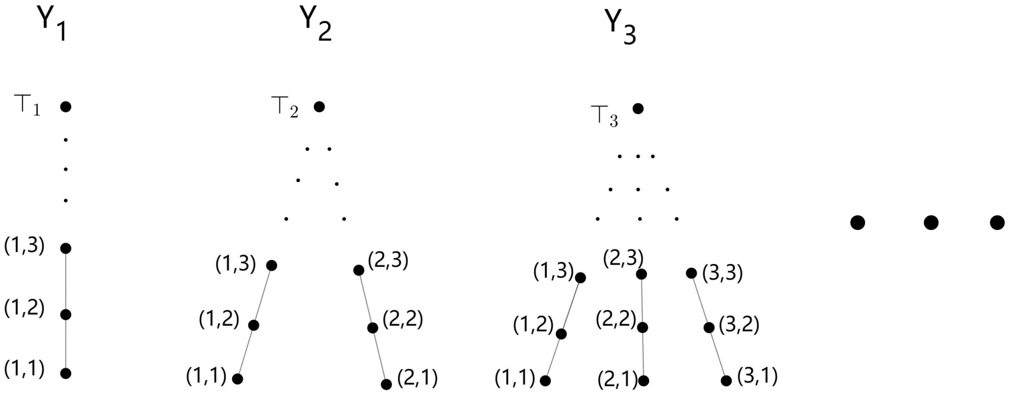
Example

There exists a strongly quasicontinuous dcpo L such that $P_L(L)$ is not strongly quasicontinuous.

L = Disjoint union of infinite strongly quasicontinuous dcpos



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Thank You!