Pointfree topology and Constructive Mathematics

Graham Manuell
graham@manuell.me
University of Coimbra

TACL Summer School
June 2022

Topology

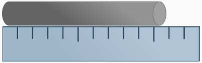
An example

Consider a rod: 

What can we say for certain about its length ℓ?

We *cannot* tell that $\ell = 10\,\text{cm}$.

Any measurement will have finite precision:



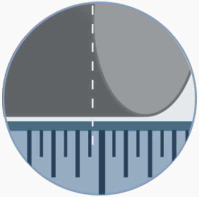
Here maybe we measure $10.1 \pm 0.3\,\text{cm}$.

But we can never be sure it is exactly $10\,\text{cm}$ long.

An example

What *can* we say?

We can tell that $\ell < 10\,\text{cm}$ if an even more sensitive measurement gives $9.91 \pm 0.02\,\text{cm}$.



This is related to the interval $(0, 10)$ being an open subset of the space $\mathbb{R}^+$ of all possible lengths of the rod.

The open sets $U \subseteq \mathbb{R}^+$ are *precisely* the sets such that if $\ell \in U$ we can verify this by finite means. (If $\ell \notin U$, who knows.)

Opens as verifiable properties

So opens can be understood as verifiable properties and topology as the study of these properties.

- The property that is always true is verifiable.
- If U and V are a verifiable properties, then so is $U \wedge V$.
- If $\mathcal{U}$ is a set of verifiable properties, then $\bigvee \mathcal{U}$ is verifiable.

Thus, this idea motivates the axioms of a topological space.

We call the logic of verifiability geometric logic.

Let $f \colon X \to Y$ be a function we could implement in the real world. Then we can verify that $x \in f^{-1}(U)$ by verifying $f(x) \in U$.

This motivates the definition continuous maps between spaces.

Topological spaces identify verifiable properties with the sets of things that satisfy them.

It is also possibly to study them more abstractly.

Definition

A frame is a poset with finite meets and arbitrary joins satisfying the distributivity condition $a \wedge \bigvee_\alpha b_\alpha = \bigvee_\alpha a \wedge b_\alpha$.

The lattice of opens of any topological space gives a frame.

Points

In the abstract approach we no longer have explicit 'points' that can satisfy the verifiable properties.

From a logical perspective points are models of the theory of verifiable properties given by the frame.

Here a model is a consistent assignment of truth values to each verifiable property.

- The top element 1 must hold in our model.
- If a holds and $a \leq b$, then b should hold.
- If a and b hold, so should $a \wedge b$.
- If $\bigvee_\alpha a_\alpha$ holds, then a_α should hold for some α.

We call such an assignment a point of the frame.

Frame homomorphisms

Frames are algebraic structures (with operations 1, $\wedge$ and a proper class of join operations of various arities).

A frame homomorphism is a map between frames that preserves these operations.

Continuous maps between topological spaces give frame homomorphisms between their frames of opens *in the opposite direction*!

We define the category of locales $\mathbf{Loc}$ to be the *opposite* of the category of frames $\mathbf{Frm}$.

We write $\mathcal{O}X$ for the frame corresponding to a locale X and f^* for the frame homomorphism corresponding to a locale map f.

Locales vs spaces

We have seen how a topological space gives rise to a locale. This gives a functor from $\mathbf{Top}$ to $\mathbf{Loc}$.

This has a right adjoint which sends a locale X to its set of points equipped with the obvious topology.

Let's look at morphisms: suppose $f\colon X \to Y$ is a locale map. Consider a point of X defined by the opens $P \subseteq \mathcal{O}X$ being 'true'. Then $\{a \in \mathcal{O}Y \mid f^*(a) \in P\}$ gives a point of Y.

This adjunction is idempotent. Locales coming from spaces are called spatial and spaces coming from locales are called sober.

Constructive mathematics

Logics

Classical logic is concerned with a Platonic notion of *truth*.

But there are other things logics can describe:

- What we already *know* to be true
- Computability
- Local truth — *where* statements are true
- Verifiable truths
- Probabilities, possibilities, fuzzy concepts, resources, etc.

An example — local truth

Suppose we have some proposition that varies with location.

For example, “Is the temperature greater than 0 °C?”

The answer to this question is not just ‘yes’ or ‘no’, but the *region* in which the proposition is true.

We will suppose this region is an open set in some fixed space.

An example — local truth

Here the usual logic constants and connectives take on new meanings.

- $\top$ means true *everywhere*
- $\bot$ means true *nowhere*
- $\wedge$ means intersection of regions
- $\vee$ means union of regions

The meaning of negation is particularly subtle. When U is open the complement U^c is seldom open. Instead we use $\operatorname{int}(U^c)$.

But the union $U \cup \operatorname{int}(U^c)$ is not the whole space if U is not clopen.

So the principle of excluded middle $p \vee \neg p$ fails for local truth!

Intuitionistic logic

Intuitionistic logic is logic ‘without the principle of excluded middle’ (or equivalent statements like $\neg\neg p \implies p$).

Without excluded middle implication cannot be defined from negation, so we need it as a basic connective.
(Negation $\neg p$ can still be defined as $p \Rightarrow \bot$.)

Algebraically, propositional intuitionistic logic is interpreted in Heyting algebras instead of Boolean algebras.

Definition

A *Heyting algebra* is a lattice with an operation $\Rightarrow$ satisfying

$$a \leq b \Rightarrow c \iff a \wedge b \leq c.$$

Intuitionistic logic versus geometric logic

In a frame, $b \wedge (-)$ preserves joins and so has an adjoint $b \Rightarrow (-)$. Thus, frames are Heyting algebras.

However, frame homomorphisms do not need to preserve $\Rightarrow$.
On the other hand, Heyting algebras needn't have infinite joins.

Geometric logic is still important for topology. In topology complements of open sets give something new: closed sets.

Intuitionistic logic is more expressive, since we can use implication and also quantifiers $(\exists, \forall)$ and higher-order logic.

The lattice of truth values

Classically there are exactly two truth values: $\bot$, $\top$.

Constructively, there is still a lattice of truth values Ω.

If $p \in \Omega$, then $p = \top$ iff p holds and $p = \bot$ iff $\neg p$ holds.
So $(p = \top) \vee (p = \bot)$ is an equivalent to excluded middle.
But we still have, $p \neq \top \implies p = \bot$.

If X is a set and $\chi \colon X \to \Omega$, then $\{x \in X \mid \chi(x) = \top\}$ is a subset of X.
Conversely, if $S \subseteq X$ then we can define a map $x \mapsto [\![x \in S]\!]$.
These are inverses. So Ω^X is isomorphic to the powerset of X.

The frame of truth values

The lattice Ω is a *frame*.

Joins exist since for $S \subseteq \Omega$, $\bigvee S = [\![\top \in S]\!]$.

In fact, Ω is the initial frame!

Let L be a frame. The unique map $! \colon \Omega \to L$ sends p to $\bigvee\{1 \mid p\}$.

Frame homomorphisms $h \colon L \to \Omega$ correspond to *points* of L.

In Loc these are maps from 1 as we might expect.

Decidable propositions

The set Ω is larger than $2 = \{\bot, \top\}$ in general, but the latter still has a role to play.

The elements of elements of 2^X correspond to decidable subsets of X — subsets $S \subseteq X$ such that $x \in S \vee x \notin S$.

Decidable subsets are analogous to clopen subsets in topology (or complemented elements in a frame).

Equality is a decidable relation on the set of natural numbers $\mathbb{N}$ and the rationals $\mathbb{Q}$. So $\forall n, m \in \mathbb{N}.\ n = m \vee n \neq m$.

Why work constructively?

If results are proved constructively they hold more generally than classical results do.

Topos	Interpretation	Principles allowed
Set	Classical results	Axiom of Choice
G-Set	G-equivariant topology	Excluded middle
Eff	Computable analysis	Dependent choice
$\mathsf{Sh}(B)$	Fibrewise topology over B	–

Presentations and classifying locales

Frame presentations

Since frames are algebraic structures, we can present them by generators and relations.

Consider the presentation $\langle g_1, g_2, \cdots \mid g_1 \wedge g_2 \leq g_3,\ g_4 \leq \bigvee_{i=5}^{\infty} g_i \rangle$.

Homomorphisms from this frame to another frame L are uniquely defined by given an element $\overline{g_i}$ of L for each generator g_i, where we must check that the $\overline{g_i}$'s satisfy the necessary relations in L.

In particular, *points* of this frame correspond to subsets S of $\{g_i \mid i \in \mathbb{Z}^+\}$ such that

- $g_1 \in S \wedge g_2 \in S \implies g_3 \in S$,
- $g_4 \in S \implies \exists i \geq 5.\ g_i \in S$.

Geometric theories

Presentations can be understood as defining geometric theories.

- The generators give basic propositions.
- The relations give axioms.

Recall that points are models of the theory. A model tells us which propositions are true!

Geometric definitions of the points are actually enough to define the topology.

Example — the reals

The real numbers $\mathbb{R}$ can be constructed by Dedekind cuts.

A Dedekind cut is a pair (L, U) of sets of rational numbers.
They satisfy the following axioms.

- If $p \leq q$ and $q \in L$ then $p \in L$ (L is downward closed)
- If $p \in L$ then $q \in L$ for some $q > p$ (L is rounded)
- There is some $q \in L$ (L is inhabited)
- If $p \leq q$ and $p \in U$ then $q \in U$ (U is upwards closed)
- If $q \in U$ then $p \in U$ for some $p < q$ (U is rounded)
- There is some $q \in U$ (U is inhabited)
- If $p \in L$ and $q \in U$ then $p < q$ (L and U are disjoint)
- If $p < q$ then either $p \in L$ or $q \in U$ (locatedness)

Example — the reals

This has the form of a geometric theory!

We have a basic proposition ℓ_q for each $q \in \mathbb{Q}$ — think "$q \in L$", and a basic proposition u_q for each $q \in \mathbb{Q}$ — think "$q \in U$".

The axioms give:

- $\ell_q \vdash \ell_p$ for $p \leq q$
- $\ell_p \vdash \bigvee_{q>p} \ell_q$ for $p \in \mathbb{Q}$
- $\top \vdash \bigvee_{q\in\mathbb{Q}} \ell_q$
- $u_p \vdash u_q$ for $p \leq q$
- $u_q \vdash \bigvee_{p<q} u_p$ for $q \in \mathbb{Q}$
- $\top \vdash \bigvee_{q\in\mathbb{Q}} u_q$
- $\ell_p \wedge u_q \vdash [\![q < p]\!]$ for $q, p \in \mathbb{Q}$
- $\top \vdash \ell_p \vee u_q$ for $p < q$

Example — the reals

Combining some of these relations together we arrive at

$$
\begin{aligned}
\mathcal{O}\mathbb{R} = \langle \ell_q, u_q,\ q \in \mathbb{Q} \mid \ell_p = \bigvee_{q>p} \ell_q,\ u_q = \bigvee_{p<q} u_p,& \\
\bigvee_{q\in\mathbb{Q}} \ell_q = 1,\ \bigvee_{q\in\mathbb{Q}} u_q = 1,& \\
\ell_p \wedge u_q = 0 \ \text{ for } p \geq q,& \\
\ell_p \vee u_q = 1 \ \text{ for } p < q \rangle&
\end{aligned}
$$

The generator ℓ_q corresponds to the open interval (q, ∞) and the generator u_q corresponds to the open interval $(-\infty, q)$.

From just the geometric definition of the points we have obtained the entire locale of reals!

Example — Cantor space

The points of Cantor space $2^{\mathbb{N}}$ are infinite sequences of bits 0 or 1.

We can verify if the n^{th} element of the sequence is 0 and 1, giving generators z_n and u_n.

This suggests $\mathcal{O}(2^{\mathbb{N}}) \cong \langle z_n, u_n,\ n \in \mathbb{N} \mid z_n \wedge u_n = 0,\ z_n \vee u_n = 1 \rangle$.

The points correspond to the decidable subsets of $\mathbb{N}$ as we expect.

Example — Sierpiński space

Consider the frame $\langle g \rangle$ with one generator and no relations.

Points correspond to truth values.

This is the frame of opens of Sierpiński space: the set of points is Ω and the topology is generated by the single subbasic open $\{\top\}$.

More generally, the points of the free frame on G generators will be given by subsets of G. The space is homeomorphic to $\mathbb{S}^G$.

Example — the Stone spectrum of a distributive lattice

Let L be a bounded distributive lattice. The Stone spectrum of L is the space of prime filters of L.

A prime filter is a subset $F \subseteq L$ such that

- if $a \leq b$ and $a \in F$ then $b \in F$,
- $1 \in F$,
- if $a \in F$ and $b \in F$ then $a \wedge b \in F$,
- $0 \notin F$,
- if $a \vee b \in F$ then $a \in F$ or $b \in F$.

This gives the presentation

$$\langle \overline{a},\ a \in L \mid \overline{1} = 1, \overline{a} \wedge \overline{b} = \overline{a \wedge b}, \overline{0} = 0, \overline{a} \vee \overline{b} = \overline{a \vee b} \rangle.$$

where $\overline{a}$ is a basic proposition asserting that a lies in the filter.

Example — surjections from $\mathbb{N}$ to X

Fix a set X and consider the following geometric theory.

Basic propositions are denoted by $[f(n) = x]$ for $n \in \mathbb{N}$ and $x \in X$.

- $[f(n) = x] \wedge [f(n) = y] \vdash [\![x = y]\!]$ for $x, y \in X$,
- $\top \vdash \bigvee_{x \in X} [f(n) = x]$ for $n \in \mathbb{N}$,
- $\top \vdash \bigvee_{n \in \mathbb{N}} [f(n) = x]$ for $x \in X$.

The points correspond to surjections from $\mathbb{N}$ to X.

If X is chosen to be large enough there are no such surjections!

However, it is still a nontrivial locale.