Pointfree topology and Constructive Mathematics

Graham Manuell
graham@manuell.me
University of Coimbra

TACL Summer School
June 2022

Remember from yesterday: Congruences and sublocales

A presentation $\langle G \mid R \rangle$ is then given the quotient of the free frame on G by the congruence generated by R.

Frame *quotients* correspond to *sublocales*.

For every $a \in \mathcal{O}X$ there is an open sublocale of X with congruence

$$\Delta_a = \langle (a, 1) \rangle = \{(u, v) \mid u \wedge a = v \wedge a\}$$

and a closed sublocale with congruence

$$\nabla_a = \langle (0, a) \rangle = \{(u, v) \mid u \vee a = v \vee a\}.$$

These are complements in the lattice of sublocales.

Moreover, we have $\mathcal{O}X/\Delta_a \cong {\downarrow} a$ and $\mathcal{O}X/\nabla_a \cong {\uparrow} a$.

Lemma

The open and closed sublocales induced by a are mutual complements in the lattice of sublocales.

Proof.

Firstly, $\nabla_a \vee \Delta_a = \langle(0, a)\rangle \vee \langle(a, 1)\rangle \supseteq \langle(0, 1)\rangle = L \times L$. Now take $(u, v) \in \nabla_a \cap \Delta_a$. Consider $(u \wedge v) \vee (u \wedge a) = u \wedge (v \vee a)$. Since $(u, v) \in \nabla_a$, we know $v \vee a = u \vee a$. So $u \wedge (v \vee a) = u$. Similarly, $(u \wedge v) \vee (v \wedge a) = v$. But also $u \wedge a = v \wedge a$ and so these agree. $\square$

Open sublocales of discrete locales

A set X can be viewed as a space with the discrete topology.
The frame of opens of the space X is just the powerset Ω^X.

Lemma

Discrete spaces are sober — all points of Ω^X come from points of X.

Proof.

The points of Ω^X are filters $\mathcal{F}$ such that if $\bigcup \mathcal{S} \in \mathcal{F}$ then $\exists S \in \mathcal{S} \cap \mathcal{F}$.

But $\bigcup_{x \in X}\{x\} = X \in \mathcal{F}$ and so $\{x\} \in \mathcal{F}$ for some $x \in X$. Now suppose $U \in \mathcal{F}$. Then $U \cap \{x\} \in F$. But $U \cap \{x\} = \bigcup\{\{x\} \text{ (fixed)} \mid x \in U\} \in \mathcal{F}$. Thus, this set is inhabited and $x \in U$. So $\mathcal{F} = \{U \subseteq X \mid x \in U\}$. $\square$

Open sublocales of the discrete locale X are simply *subsets*.
Opens are, of course, subsets $S \subseteq X$ and $\downarrow S \cong \Omega^S$.

Closed sublocales of discrete locales

Since closed sublocales and open sublocales are complements, sublocales of discrete spaces which are both closed and open correspond to decidable subsets.

Closed sublocales can be specified by their complementary subsets. This explains some strange definitions in constructive algebra.

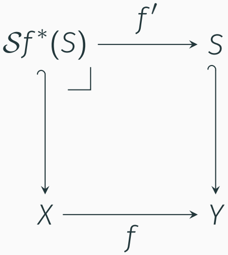
Definition

Let R be a ring. An anti-ideal of R is a subset $A \subseteq R$ such that

- $0 \notin A$,
- if $x + y \in A$ then $x \in A \vee y \in A$,
- If $xy \in A$ then $x \in A \wedge y \in A$.

Preimages of sublocales

We can easily define preimages of sublocales categorically.



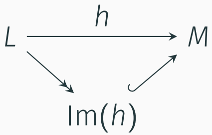
This makes the sublocale lattice into a functor $\mathcal{S}\colon \mathrm{Loc}^{\mathrm{op}} \to \mathrm{Pos}$.

By standard algebraic arguments, if C_S is the congruence for S then the congruence corresponding to $\mathcal{S}f^*(S)$ is $\langle (f^* \times f^*)(C_S) \rangle$.

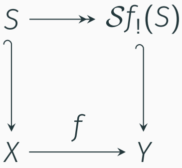
Note that preimages of open/closed sublocales are open/closed.

Images of sublocales

Like for any algebraic structure, frame homomorphisms have image factorisations.



Thus a locale map $f\colon X \to Y$ factorises into an epimorphism followed by a sublocale inclusion. We can use this to define images of sublocales.



We have $C_{\mathcal{S}f_!(S)} = (f^* \times f^*)^{-1}(C_S)$. The map $\mathcal{S}f_!$ is left adjoint to $\mathcal{S}f^*$.

Products of locales

Coproducts of frames

Products of locales correspond to *coproducts* of their frames.

These coproducts can be computed as in any algebraic structure.

The coproduct $L \oplus M$ can be presented by generators $\{\iota_1(\ell) \mid \ell \in L\} \sqcup \{\iota_2(m) \mid m \in M\}$ and relations making the maps $\iota_{1,2}$ frame homomorphisms.

Define $\ell \oplus m = \iota_1(\ell) \wedge \iota_2(m)$. These elements correspond to (basic) open rectangles in the product topology.

More generally, $L \cong \langle G^L \mid R^L \rangle$ and $M \cong \langle G^M \mid R^M \rangle$ then

$$L \oplus M \cong \langle G^L \sqcup G^M \mid R^L \sqcup R^M \rangle.$$

Discrete locales

Recall: a set X can be viewed as a discrete locale with frame Ω^X. Basic opens are given by singletons $\{x\}$.

Lemma

The binary products of discrete locales agree with the product of the underlying sets.

Proof.

The coproduct frame $\Omega^X \oplus \Omega^Y$ has basic opens $\{x\} \oplus \{y\}$. The points are given by $\mathrm{Hom}(\Omega^X \oplus \Omega^Y, \Omega) \cong \mathrm{Hom}(\Omega^X, \Omega) \times \mathrm{Hom}(\Omega^Y, \Omega) \cong X \times Y$.

To show $\Omega^X \oplus \Omega^Y \cong \Omega^{X \times Y}$ is suffices to show that opens are distinguished by the points by contain. The open $u = \bigvee_\alpha S_\alpha \oplus T_\alpha$ contains the points (x, y) for which $x \in S_\alpha$ and $y \in T_\alpha$ for some α.

But $x \in S_\alpha$ iff $\{x\} \in S_\alpha$ and so $(x, y) \in U$ iff $\{x\} \oplus \{y\} \leq u$, and we know the basic opens contained in an open determine it. □

Hausdorffness

Definition

A locale X is Hausdorff if the diagonal in $X \times X$ is closed.

According to our intuition this means equality is refutable — that is, inequality is verifiable.

In terms of the frames, the codiagonal map is $\Delta^* \colon u \oplus v \mapsto u \wedge v$. This is clearly surjective. It being closed means $u \wedge v = u' \wedge v' \iff (u \oplus v) \vee a = (u' \oplus v') \vee a$ for some $a \in \mathcal{O}X \oplus \mathcal{O}X$. In fact, we must have $a = \bigvee\{u \oplus v \mid u \wedge v = 0\}$, the largest element that Δ^* maps to 0. (Then the backward implication is automatic.) It suffices to show $u \oplus v \leq (u \wedge v) \oplus (u \wedge v) \vee a$ (with a as above).

It is not hard to see that sublocales and products of Hausdorff locales are Hausdorff.

The reals are Hausdorff

Recall the presentation of the locale of reals.

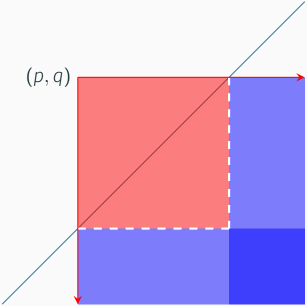
$$\begin{aligned}\mathcal{O}\mathbb{R} = \langle (\!(q, \infty)\!), (\!(-\infty, q)\!),\ q \in \mathbb{Q} \mid \\ (\!(p, \infty)\!) = \bigvee_{q>p} (\!(q, \infty)\!),\ (\!(-\infty, q)\!) = \bigvee_{p<q} (\!(-\infty, p)\!), \\ \bigvee_q (\!(q, \infty)\!) = 1,\ \bigvee_q (\!(-\infty, q)\!) = 1, \\ (\!(-\infty, q)\!) \wedge (\!(p, \infty)\!) = 0 \text{ for } p \geq q, \\ 1 \leq (\!(p, \infty)\!) \vee (\!(-\infty, q)\!) \text{ for } p < q \rangle.\end{aligned}$$

The putative diagonal complement is
$d = \bigvee_r (\!(-\infty, r)\!) \oplus (\!(r, \infty)\!) \vee \bigvee_r (\!(r, \infty)\!) \oplus (\!(-\infty, r)\!)$.

To show Hausdorffness there are a few cases, but a representative one is $(\!(p, \infty)\!) \oplus (\!(-\infty, q)\!) \leq (\!(p, q)\!) \oplus (\!(p, q)\!) \vee d$.

The reals are Hausdorff

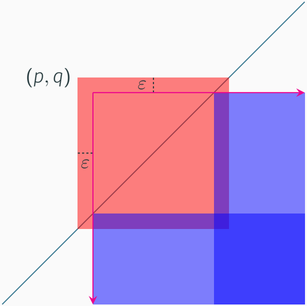
We must show $(\!(p, \infty)\!) \oplus (\!(-\infty, q)\!) \leq (\!(p, q)\!) \oplus (\!(p, q)\!) \vee d$.



This diagram suggests trying $(\!(p, \infty)\!) \oplus (\!(-\infty, q)\!) \leq$
$(\!(p, q)\!) \oplus (\!(p, q)\!) \vee (\!(p, \infty)\!) \oplus (\!(-\infty, p)\!) \vee (\!(q, \infty)\!) \oplus (\!(-\infty, q)\!)$.

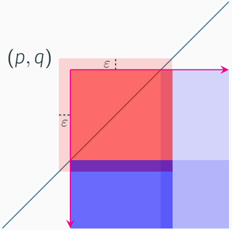
The reals are Hausdorff

We must show $(\!(p, \infty)\!) \oplus (\!(-\infty, q)\!) \leq (\!(p, q)\!) \oplus (\!(p, q)\!) \vee d$.



But this suggests we do have $(\!(p+\varepsilon, \infty)\!) \oplus (\!(-\infty, q-\varepsilon)\!) \leq (\!(p, q)\!) \oplus (\!(p, q)\!) \vee (\!(p+\varepsilon, \infty)\!) \oplus (\!(-\infty, p+\varepsilon)\!) \vee (\!(q-\varepsilon, \infty)\!) \oplus (\!(-\infty, q-\varepsilon)\!)$.

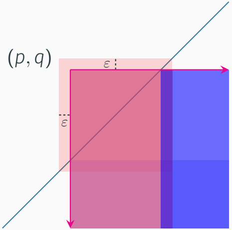
The reals are Hausdorff



First consider

$$
\begin{aligned}
&(\!(p+\varepsilon, q)\!) \oplus (\!(p, q-\varepsilon)\!) \vee (\!(p+\varepsilon, q)\!) \oplus (\!(-\infty, p+\varepsilon)\!) \\
=\ &(\!(p+\varepsilon, q)\!) \oplus [(\!(p, q-\varepsilon)\!) \vee (\!(-\infty, p+\varepsilon)\!)] \\
=\ &(\!(p+\varepsilon, q)\!) \oplus [(\!(p, \infty)\!) \wedge (\!(-\infty, q-\varepsilon)\!) \vee (\!(-\infty, p+\varepsilon)\!)] \\
=\ &(\!(p+\varepsilon, q)\!) \oplus [((\!(p, \infty)\!) \vee (\!(-\infty, p+\varepsilon)\!)) \wedge (\!(-\infty, q-\varepsilon)\!)] \\
=\ &(\!(p+\varepsilon, q)\!) \oplus (\!(-\infty, q-\varepsilon)\!).
\end{aligned}
$$

The reals are Hausdorff



Now we have

$$
\begin{aligned}
&(\!(p+\varepsilon, q)\!) \oplus (\!(-\infty, q-\varepsilon)\!) \vee (\!(q-\varepsilon, \infty)\!) \oplus (\!(-\infty, q-\varepsilon)\!) \\
= &\,[(\!(p+\varepsilon, q)\!) \vee (\!(q-\varepsilon, \infty)\!)] \oplus (\!(-\infty, q-\varepsilon)\!) \\
= &\,[(\!(p+\varepsilon, \infty)\!) \wedge (\!(-\infty, q)\!) \vee (\!(q-\varepsilon, \infty)\!)] \oplus (\!(-\infty, q-\varepsilon)\!) \\
= &\,[(\!(p+\varepsilon, \infty)\!) \wedge ((\!(-\infty, q)\!) \vee (\!(q-\varepsilon, \infty)\!))] \oplus (\!(-\infty, q-\varepsilon)\!) \\
= &\,(\!(p+\varepsilon, \infty)\!) \oplus (\!(-\infty, q-\varepsilon)\!).
\end{aligned}
$$

The reals are Hausdorff

In summary, we have shown

$$\begin{aligned} &((p+\varepsilon, q)) \oplus ((p, q-\varepsilon)) \vee ((p+\varepsilon, q)) \oplus ((-\infty, p+\varepsilon)) \\ &\qquad \vee ((q-\varepsilon, \infty)) \oplus ((-\infty, q-\varepsilon)) \\ &= ((p+\varepsilon, q)) \oplus ((-\infty, q-\varepsilon)) \vee ((q-\varepsilon, \infty)) \oplus ((-\infty, q-\varepsilon)) \\ &= ((p+\varepsilon, \infty)) \oplus ((-\infty, q-\varepsilon)). \end{aligned}$$

Thus, $((p+\varepsilon, \infty)) \oplus ((-\infty, q-\varepsilon)) \leq ((p, q)) \oplus ((p, q)) \vee d$.

Now taking the join over all sufficiently small $\varepsilon > 0$ we have

$$\begin{aligned} ((p, q)) \oplus ((p, q)) \vee d &\geq \bigvee_{\varepsilon} ((p+\varepsilon, \infty)) \oplus ((-\infty, q-\varepsilon)) \\ &\geq \bigvee_{\varepsilon, \varepsilon'} ((p+\varepsilon, \infty)) \oplus ((-\infty, q-\varepsilon')) \\ &= \bigvee_{\varepsilon} \iota_1(((p+\varepsilon, \infty))) \wedge \bigvee_{\varepsilon'} \iota_2(((-\infty, q-\varepsilon'))) \\ &= ((p, \infty)) \oplus ((-\infty, q)). \end{aligned}$$

□

Diagonals of discrete locales

Let X be a discrete locale. Since the diagonal of $X \times X$ is a subset it is *open*. So in discrete locales equality is *verifiable*.

If the diagonal is also *closed*, this means it has a complement and thus is a decidable subset.

So for a discrete locale to be Hausdorff means it has decidable equality.

Compactness and overtness

Verifying universal quantification

Recall the verifiability interpretation of topology.

Suppose we have an open $U \subseteq X \times Y$.
Can we verify when $y \in Y$ satisfies $\forall x \in X.\ (x, y) \in U$?

This is easy if X is a finite set — just check each $(x_i, y) \in U$ in turn.

For an infinite set X this would appear to be impossible.

However, there *are* other locales X which act like (Kuratowski-) finite sets in this regard!

These will turn out to be the compact locales.

Verifying universal quantification

In set-theoretic terms, we are asking if the set $\{y \in Y \mid \forall x \in X.\ (x, y) \in U\}$ is open.

Taking the classical complement, we are asking if the set $\{y \in Y \mid \exists x \in X.\ (x, y) \notin U\}$ is closed.

This is just the image of the closed set U^{c} under the projection $\pi_2 \colon X \times Y \to Y$.

Thus, the universal quantification over a locale X of a verifiable property is verifiable whenever the images of closed sublocales under $\pi_2 \colon X \times Y \to Y$ are closed (for all Y).

Verifying existential quantification

Suppose we have an open $U \subseteq X \times Y$.
Can we verify when $y \in Y$ satisfies $\exists x \in X.\ (x, y) \in U$?

If X is a set (a discrete locale) then we can just take an $x \in X$ that works and verify $(x, y) \in U$.

More explicitly, we are asking if $\{y \in Y \mid \exists x \in X.\ (x, y) \in U\}$ is open.
This is just the image of U under the projection $\pi_2 \colon X \times Y \to Y$.

Thus, the existential quantification over a locale X of a verifiable property is verifiable whenever the images of open sublocales under $\pi_2 \colon X \times Y \to Y$ are open (for all Y).

This property is called overtness and is *dual* to compactness.

Another definition of overtness

Definition

An element a in a frame $\mathcal{O}X$ is said to be positive (written $a > 0$) if $a \leq \bigvee S$ implies S is inhabited.

Definition

A locale X is overt if $\mathcal{O}X$ has a base of positive elements.

Classically, $a > 0 \iff a \neq 0$ and so *every* locale is overt!

Discrete locales are overt since singletons form a positive base.

The locale $\mathbb{R}$ of reals is overt since the elements $(\!(p, q)\!)$ for $p < q$ form a base. These are positive since intuitively any cover of them is built from the basic relations in which all the nontrivial joins are inhabited.

Open morphisms

Definition

A locale map $f\colon X \to Y$ is open if $\mathcal{S}f_!$ maps open sublocales to open sublocales.

Lemma

A map $f\colon X \to Y$ is open if and only if $f^*\colon \mathcal{O}Y \to \mathcal{O}X$ has a left adjoint $f_!\colon \mathcal{O}X \to \mathcal{O}Y$ satisfying $f_!(f^*(b) \wedge a) = b \wedge f_!(a)$.

Proof.

($\Longrightarrow$) Let $g\colon \mathcal{O}X \to \mathcal{O}Y$ be such that $(f^* \times f^*)^{-1}(\Delta_a) = \Delta_{g(a)}$. Then $f^*(u) \wedge a = f^*(v) \wedge a \iff u \wedge g(a) = v \wedge g(a)$. Taking $u = 1$, we obtain $a \leq f^*(v) \iff g(a) \leq v$ and so $g \dashv f^*$. Now letting $v = w \wedge u$, the right-hand side becomes $u \wedge g(a) \leq w$ and the left-hand side becomes $f^*(u) \wedge a \leq f^*(w)$, which is equivalent to $f_!(f^*(u) \wedge a) \leq w$. So $f_!(f^*(u) \wedge a) = u \wedge g(a) = u \wedge f_!(a)$.

Lemma

A map $f\colon X \to Y$ is open if and only if $f^*\colon \mathcal{O}Y \to \mathcal{O}X$ has a left adjoint $f_!\colon \mathcal{O}X \to \mathcal{O}Y$ satisfying $f_!(f^*(b) \wedge a) = b \wedge f_!(a)$.

Proof.

($\Longleftarrow$) Conversely, $f_!(f^*(b) \wedge a) = b \wedge f_!(a)$ means that $f^*(b) \wedge a \leq f^*(w) \iff b \wedge f_!(a) \leq w$.

Now $x \wedge c \leq y$ precisely when $x \wedge c \leq y \wedge c$ and so $f^*(b) \wedge a \leq f^*(w) \wedge a \iff b \wedge f_!(a) \leq w \wedge f_!(a)$.

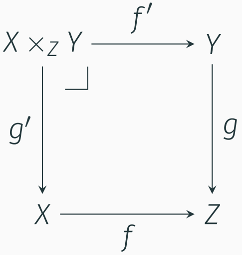
This is precisely what it means to have $(f^* \times f^*)^{-1}(\Delta_a) = \Delta_{f_!(a)}$ and so we are done. □

Pullback stability of open maps

Theorem

Open maps are stable under pullback.

Explicitly, this means that in following pullback diagram in $\mathbf{Loc}$, if g is open then so is g'.



We omit the proof.

Here $g'^*(a) = [a \oplus 1]$ and $g'_!(a \oplus b) = a \wedge f^* g_!(b)$.

Overtness

Theorem

Let X be a locale. The following are equivalent.

1. *For all Y, the projection $\pi_2 \colon X \times Y \to Y$ is open.*
2. *The unique map $! \colon X \to 1$ is open.*
3. *The frame map $!^* \colon \Omega \to \mathcal{O}X$ has a left adjoint.*
4. *The frame $\mathcal{O}X$ has a base of positive elements.*

Proof.

The implications (1) $\implies$ (2) $\implies$ (3) are obvious.

We will show (3) $\implies$ (4) and (4) $\implies$ (3) $\implies$ (2) $\implies$ (1).

Proof.

(3) $\implies$ (4) Let $\exists\colon \mathcal{O}X \to \Omega$ be the left adjoint to $!^*$. Now $\exists(a) = \top$ means $\exists(a) \leq p \implies p = \top$ and hence $a \leq !^*(p) \implies p = \top$.

Recall $!^*(p) = \bigvee\{\top \mid p = \top\}$. So if $a > 0$ this certainly holds.

On the other hand, suppose the implication holds and $a \leq \bigvee S$. Then $a \leq \bigvee\{s \mid s \in S\} \leq \bigvee\{1 \mid s \in S\} = !^*([\![\exists s \in S]\!])$ and so $\exists s \in S$ by assumption. Thus, $a > 0$. So we have shown $\exists(a) = [\![a > 0]\!]$.

Now by adjointness $a \leq !^*\exists(a) = \bigvee\{\top \mid a > 0\}$. So then $a = \bigvee\{a \mid a > 0\}$, which is a join of positive elements as required. □

Overtness

Proof.
(4) $\implies$ (3) Suppose $\mathcal{O}X$ is overt. We claim $\exists\colon a \mapsto [\![a > 0]\!]$ is a left adjoint to $!^*\colon \Omega \to \mathcal{O}X$.

It is clear that if $\bigvee\{1 \mid p = \top\} > 0$ then $p = \top$. Thus, $\exists \circ !^* \leq \mathrm{id}_\Omega$.

Now take $a \in \mathcal{O}X$ and write $a = \bigvee_\alpha a_\alpha$ where each $a_\alpha > 0$. Then $a = \bigvee\{a_\alpha \mid a_\alpha > 0\} \leq \bigvee\{1 \mid a_\alpha > 0\} \leq \bigvee\{1 \mid a > 0\} = !^*\exists(a)$.
So $\exists \dashv !^*$.

(3) $\implies$ (2) Let $\exists \dashv !^*$. We must show $\exists(!^*(p) \wedge a) = p \wedge \exists(a)$. A meet $!^*(p) \wedge a$ can be written as a *join* $\bigvee\{a \mid p = \top\}$. Since the left adjoint $\exists$ preserves joins the desired equality follows.

(2) $\implies$ (1) The projection $\pi_2\colon X \times Y \to Y$ is the pullback of the open map $!\colon X \to 1$ along $!\colon Y \to 1$. □

Preservation of overtness

Lemma

Open sublocales of overt locales are overt.

Proof.

The positive base for the frame $\downarrow a$ can be taken to be the restriction of the positive base for the parent frame. $\square$

Lemma

Images of overt (sub)locales are overt.

Proof.

It suffices to show that subframes of overt frames are overt. Let M be a subframe of an overt frame L and consider $a \in M$. In L we have $a = \bigvee\{a \mid a > 0\}$, but this join works equally well in M. $\square$

Non-overt locales

Lemma

Every overt sublocale V of a discrete locale X is open.

Proof.

Intuitively, we can verify $x \in V$ by showing $\exists y \in V.\ x = y$. More formally, consider the following pullback.

$$\begin{array}{ccc} V & \xrightarrow{(i, \mathrm{id})} & X \times V \\ {\scriptstyle i}\downarrow & \lrcorner & \downarrow{\scriptstyle X \times i} \\ X & \xrightarrow[(\mathrm{id}, \mathrm{id})]{} & X \times X \end{array}$$

Since the diagonal $(\mathrm{id}, \mathrm{id})$ is open, so is (i, id). Since V is overt, $\pi_1 \colon X \times V \to X$ is open. Thus, the composite $i = \pi_1(i, \mathrm{id})$ is open. □

Corollary

If every closed sublocale of 1 *is overt, then excluded middle holds.*