Pointfree topology and Constructive Mathematics

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Remember from yesterday: Overtness and compactness

If for every Y and every verifiable property $U \in \mathcal{O}(X \times Y)$ we have a verifiable property corresponding to $\{y \in Y \mid \exists x \in X.\ (x, y) \in X\}$, we say X is overt.

Formally, this means $\pi_2 \colon X \times Y \to Y$ is open for all Y.

If for every Y and every verifiable property $U \in \mathcal{O}(X \times Y)$ we have a verifiable property corresponding to $\{y \in Y \mid \forall x \in X.\ (x, y) \in X\}$, we say X is compact.

Formally, this means $\pi_2 \colon X \times Y \to Y$ is closed for all Y.

Open morphisms

Definition
A locale map $f\colon X \to Y$ is open if $\mathcal{S}f_!$ maps open sublocales to open sublocales.

Lemma
A map $f\colon X \to Y$ is open if and only if $f^*\colon \mathcal{O}Y \to \mathcal{O}X$ has a left adjoint $f_!\colon \mathcal{O}X \to \mathcal{O}Y$ satisfying $f_!(f^*(b) \wedge a) = b \wedge f_!(a)$.

Proof.
($\Longrightarrow$) Let $g\colon \mathcal{O}X \to \mathcal{O}Y$ be such that $(f^* \times f^*)^{-1}(\Delta_a) = \Delta_{g(a)}$. Then $f^*(u) \wedge a = f^*(v) \wedge a \iff u \wedge g(a) = v \wedge g(a)$. Taking $u = 1$, we obtain $a \leq f^*(v) \iff g(a) \leq v$ and so $g \dashv f^*$. Now letting $v = w \wedge u$, the right-hand side becomes $u \wedge g(a) \leq w$ and the left-hand side becomes $f^*(u) \wedge a \leq f^*(w)$, which is equivalent to $f_!(f^*(u) \wedge a) \leq w$. So $f_!(f^*(u) \wedge a) = u \wedge g(a) = u \wedge f_!(a)$.

Lemma
A map $f\colon X \to Y$ is open if and only if $f^*\colon \mathcal{O}Y \to \mathcal{O}X$ has a left adjoint $f_!\colon \mathcal{O}X \to \mathcal{O}Y$ satisfying $f_!(f^*(b) \wedge a) = b \wedge f_!(a)$.

Proof.
($\Longleftarrow$) Conversely, $f_!(f^*(b) \wedge a) = b \wedge f_!(a)$ means that $f^*(b) \wedge a \leq f^*(w) \iff b \wedge f_!(a) \leq w$.

Now $x \wedge c \leq y$ precisely when $x \wedge c \leq y \wedge c$ and so $f^*(b) \wedge a \leq f^*(w) \wedge a \iff b \wedge f_!(a) \leq w \wedge f_!(a)$.

This is precisely what it means to have $(f^* \times f^*)^{-1}(\Delta_a) = \Delta_{f_!(a)}$ and so we are done. $\square$

Pullback stability of open maps

Theorem

Open maps are stable under pullback.

Explicitly, this means that in following pullback diagram in Loc, if g is open then so is g'.

$$\begin{tikzcd} X \times_Z Y \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

We omit the proof.
Here $g'^*(a) = [a \oplus 1]$ and $g'_!(a \oplus b) = a \wedge f^* g_!(b)$.

Theorem

Let X be a locale. The following are equivalent.

1. *For all Y, the projection $\pi_2 \colon X \times Y \to Y$ is open.*
2. *The unique map $! \colon X \to 1$ is open.*
3. *The frame map $!^* \colon \Omega \to \mathcal{O}X$ has a left adjoint.*
4. *The frame $\mathcal{O}X$ has a base of positive elements.*

Proof.

The implications (1) $\implies$ (2) $\implies$ (3) are obvious.

We will show (3) $\implies$ (4) and (4) $\implies$ (3) $\implies$ (2) $\implies$ (1).

Proof.

(3) $\implies$ (4) Let $\exists\colon \mathcal{O}X \to \Omega$ be the left adjoint to $!^*$. Now $\exists(a) = \top$ means $\exists(a) \leq p \implies p = \top$ and hence $a \leq !^*(p) \implies p = \top$.

Recall $!^*(p) = \bigvee\{\top \mid p = \top\}$. So if $a > 0$ this certainly holds.

On the other hand, suppose the implication holds and $a \leq \bigvee S$. Then $a \leq \bigvee\{s \mid s \in S\} \leq \bigvee\{1 \mid s \in S\} = !^*(\llbracket \exists s \in S \rrbracket)$ and so $\exists s \in S$ by assumption. Thus, $a > 0$. So we have shown $\exists(a) = \llbracket a > 0 \rrbracket$.

Now by adjointness $a \leq !^*\exists(a) = \bigvee\{\top \mid a > 0\}$. So then $a = \bigvee\{a \mid a > 0\}$, which is a join of positive elements as required. $\square$

Proof.

(4) $\implies$ (3) Suppose $\mathcal{O}X$ is overt. We claim $\exists\colon a \mapsto [\![a > 0]\!]$ is a left adjoint to $!^*\colon \Omega \to \mathcal{O}X$.

It is clear that if $\bigvee\{1 \mid p = \top\} > 0$ then $p = \top$. Thus, $\exists \circ !^* \leq \mathrm{id}_\Omega$.

Now take $a \in \mathcal{O}X$ and write $a = \bigvee_\alpha a_\alpha$ where each $a_\alpha > 0$. Then $a = \bigvee\{a_\alpha \mid a_\alpha > 0\} \leq \bigvee\{1 \mid a_\alpha > 0\} \leq \bigvee\{1 \mid a > 0\} = !^*\exists(a)$. So $\exists \dashv !^*$.

(3) $\implies$ (2) Let $\exists \dashv !^*$. We must show $\exists(!^*(p) \wedge a) = p \wedge \exists(a)$. A meet $!^*(p) \wedge a$ can be written as a *join* $\bigvee\{a \mid p = \top\}$. Since the left adjoint $\exists$ preserves joins the desired equality follows.

(2) $\implies$ (1) The projection $\pi_2\colon X \times Y \to Y$ is the pullback of the open map $!\colon X \to 1$ along $!\colon Y \to 1$. $\square$

Preservation of overtness

Lemma
Open sublocales of overt locales are overt.

Proof.
The positive base for the frame $\downarrow a$ can be taken to be the restriction of the positive base for the parent frame. $\square$

Lemma
Images of overt (sub)locales are overt.

Proof.
It suffices to show that subframes of overt frames are overt. Let M be a subframe of an overt frame L and consider $a \in M$. In L we have $a = \bigvee\{a \mid a > 0\}$, but this join works equally well in M. $\square$

Preservation of overtness

Lemma

Binary products of overt locales are overt.

Proof.

Let X and Y be overt locales. Then $\pi_2 \colon X \times Y \to Y$ is open. But $! \colon Y \to 1$ is open too. Thus, the composite map $! \colon X \times Y \to Y \to 1$ is open and hence $X \times Y$ is overt. $\square$

Non-overt locales

Lemma
Every overt sublocale V of a discrete locale X is open.

Proof.
Intuitively, we can verify $x \in V$ by showing $\exists y \in V.\ x = y$. More formally, consider the following pullback.

$$\begin{tikzcd}
V \arrow[r, "{(i, \mathrm{id})}"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times V \arrow[d, "X \times i"] \\
X \arrow[r, "{(\mathrm{id}, \mathrm{id})}"'] & X \times X
\end{tikzcd}$$

Since the diagonal $(\mathrm{id}, \mathrm{id})$ is open, so is (i, id). Since V is overt, $\pi_1 \colon X \times V \to X$ is open. Thus, the composite $i = \pi_1(i, \mathrm{id})$ is open. $\square$

Corollary
If every closed sublocale of 1 *is overt, then excluded middle holds.*

Discrete locales

Proposition

A locale X is discrete if and only if it is overt and its diagonal is open.

Proof.

We have already proved the forward direction. We omit the proof of the reverse direction. □

So in a sense, discrete locales (sets) are dual to compact Hausdorff locales.

Compactness

Definition

A locale map $f\colon X \to Y$ is closed if $\mathcal{S}f_!$ maps closed sublocales to closed sublocales.

Since frame homomorphisms f^* preserve all joins, they have right adjoints f_*.

Lemma

A map $f\colon X \to Y$ is closed if and only if the frame map f^* and its right adjoint f_* satisfy $f_*(f^*(b) \vee a) = b \vee f_*(a)$.

Proof.

Omitted. $\square$

Compactness

Definition

A locale X is compact if whenever $\bigvee S = 1$ in $\mathcal{O}X$ then there is some Kuratowski-finite subset $F \subseteq S$ such that $\bigvee F = 1$.

Definition

A poset S is called directed if every Kuratowski-finite subset $F \subseteq S$ has an upper bound $b \in S$.

A locale X is compact if and only if, for every directed subset $S \subseteq \mathcal{O}X$, $\bigvee S = 1$ implies $1 \in S$.

Note that $1 = !^*(\top) \leq a \iff \top \leq !_*(a)$ and so $!_*(a) = \top \iff a = 1$. So $!^*$ preserves directed joins if and only if for S directed, $\bigvee S = 1$ implies $\exists s \in S.\ s = 1$. Thus, X is compact iff $!_*$ preserves directed joins.

Definition

A locale map $f\colon X \to Y$ is proper if it is closed and f_* preserves directed joins.

Note: unlike open maps, closed maps are *not* stable under pullback. However, proper maps are.

Theorem

Let X be a locale. The following are equivalent.

1. *For all Y, the projection $\pi_2\colon X \times Y \to Y$ is closed.*
2. *For all Y, the projection $\pi_2\colon X \times Y \to Y$ is proper.*
3. *The unique map $!\colon X \to 1$ is proper.*
4. *The right adjoint $!_*\colon \mathcal{O}X \to \Omega$ preserves directed joins.*

Preservation of compactness

Lemma
Closed sublocales of compact locales are compact.

Lemma
Images of compact (sub)locales are compact.

Lemma
Binary products of of compact locales are compact.

Examples of compact locales

Lemma

A set X is compact (as a discrete locale) iff is Kuratowski-finite.

Proof.

Suppose X is compact. We have $X = \bigcup_{x \in X} \{x\}$ and so by compactness, X is a Kuratowski-finite join of singletons. Hence X is Kuratowski-finite.

Suppose X is Kuratowski-finite. Then X is the image of some set $[n] = \{m \in \mathbb{N} \mid m < n\}$. Thus, it suffices to show $[n]$ is compact.

We proceed by induction. Certainly, $[0]$ is compact. Suppose $[n]$ is compact and consider a union $\bigcup \mathcal{S} = [n] \cup \{n\}$. Then $[n] \subseteq \bigcup \mathcal{S}$ and so there is a Kuratowski-finite subset $\mathscr{F} \subseteq \mathcal{S}$ such that $[n] \subseteq \mathscr{F}$. Moreover, $n \in S$ for some $S \in \mathcal{S}$. Thus, $\mathscr{F} \cup \{S\} \subseteq \mathcal{S}$ is a Kuratowski-finite subcover. So $[n+1]$ is compact. $\square$

Examples of compact locales

Lemma

The Cantor space $2^{\mathbb{N}}$ is compact.

Proof sketch.
We recall its presentation $\langle z_n, u_n,\ n \in \mathbb{N} \mid z_n \wedge u_n = 0,\ z_n \vee u_n = 1 \rangle$.
We need only consider covers by basic opens, z_n and u_n.

Since the presentation only uses finite joins, a general join can only equal 1 if it is “forced to by a finite join” — that is, if it has a (Kuratowski-)finite subcover. □

Lemma

The real closed interval $[0, 1]$ *is compact.*

Proof sketch.
One can show that $[0, 1]$ is an image of $2^{\mathbb{N}}$. The result follows. □

Compactness of the unit interval

When constructivists say $[0,1]$ or $2^{\mathbb{N}}$ might *not* be compact, they are referring the topological space of points.

But we have seen that the localic versions *are* compact. So in these cases they are just not *spatial*.

In particular, they are not spatial in the Effective Topos.

Andrej Bauer. *König's Lemma and Kleene Tree.* `http://math.andrej.com/wp-content/uploads/2006/05/kleene-tree.pdf`. 2006

The pointfree approach recovers classical principles constructively

Excluded middle

Constructively we do not have that every subset of a set has a complement unless excluded middle holds.

On the other hand, every subset of a set is an open sublocale and this *always* has a complementary sublocale.

Similarly, $\neg\neg p \iff p$ is not constructively valid.

On the other hand, for a truth value $p \in \Omega$, let P we the corresponding open sublocale. Then the exponential 0^P in Loc is isomorphic to the closed complement of P in 1 and $0^{0^P} \cong P$!

Steven Vickers. *Generalized point-free spaces, pointwise.* arXiv:2206.01113. 2022

Tychonoff theorem

The Tychonoff theorem for spaces is famously equivalent to the Axiom of Choice. But even constructively we have:

Theorem

Arbitrary products of compact locales are compact.

Peter Johnstone and Steven Vickers. "Preframe presentations present". In: *Category theory: Proceedings of the International Conference held in Como*. Ed. by A Carboni, M C Pedicchio, and G Rosolini. Vol. 1488. Lecture Notes in Mathematics. Berlin: Springer, 1991, pp. 193–212

The Axiom of Choice can be thought of as saying that the product of nonempty sets is nonempty. But even constructively we have:

Theorem

A product of positive overt locales indexed by a set with decidable equality is positive and overt.

Simon Henry. "Localic metric spaces and the localic Gelfand duality". In: *Adv. Math.* 294 (2016), pp. 634–688