Translational Embeddings via Stable Canonical Rules

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We present a novel uniform method for studying modal companions of superintuitionistic deductive systems and related notions.

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- Stable canonical rules, filtration, duality
- Rule systems and logics
- Generalizable to several signatures

Overview

Notation

$\mathtt{IPC}$	Intuitionistic propositional calculus
$\mathtt{S4}$	$\mathtt{K} \oplus \Box p \to p \oplus \Box p \to \Box\Box p$
$\mathtt{Grz}$	$\mathtt{S4} \oplus \Box(\Box(p \to \Box p) \to p) \to p$
HA	Heyting algebras
MA	Modal algebras
$\mathsf{S4}, \mathsf{Grz}$	Modal algebras validating $\mathtt{S4}, \mathtt{Grz}$

The Gödel Translation

Gödel [1933] defined the following translation of intuitionistic formulae into modal formulae.

- $T(\bot) := \bot$
- $T(\top) := \top$
- $T(p) := \Box p$
- $T(\varphi \wedge \psi) := T(\varphi) \wedge T(\psi)$
- $T(\varphi \vee \psi) := T(\varphi) \vee T(\psi)$
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Theorem (McKinsey and Tarski 1944)

$$\varphi \in \mathtt{IPC} \iff T(\varphi) \in \mathtt{S4}$$

Modal companions...

Definition

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$$\begin{aligned} \tau &: \mathsf{Ext}(\mathtt{IPC}) \to \mathsf{NExt}(\mathtt{S4}) \\ \mathtt{L} &\mapsto \mathtt{S4} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\} \end{aligned}$$

$$\begin{aligned} \sigma &: \mathsf{Ext}(\mathtt{IPC}) \to \mathsf{NExt}(\mathtt{S4}) \\ \mathtt{L} &\mapsto \mathbf{Grz} \oplus \tau\mathtt{L} \end{aligned}$$

$$\begin{aligned} \rho &: \mathsf{NExt}(\mathtt{S4}) \to \mathsf{Ext}(\mathtt{IPC}) \\ \mathtt{M} &\mapsto \{\varphi : T(\varphi) \in \mathtt{M}\} \end{aligned}$$

Modal companions...

Some central results on modal companions

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Theorem (Characterization theorem, Maksimova and Rybakov 1974)

The set $\rho^{-1}(\mathtt{L})$ *of modal companions of* $\mathtt{L} \in \mathsf{Ext}(\mathtt{IPC})$ *forms an interval*

$$\rho^{-1}(\mathtt{L}) = \{\mathtt{M} \in \mathsf{NExt}(\mathtt{S4}) : \tau\mathtt{L} \subseteq \mathtt{M} \subseteq \sigma\mathtt{L}\}.$$

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Theorem (Blok-Esakia theorem, Blok 1976; Esakia 1976)

The mappings $\sigma : \mathsf{Ext}(\mathtt{IPC}) \to \mathsf{NExt}(\mathtt{S4})$ *and* $\rho : \mathsf{NExt}(\mathtt{S4}) \to \mathsf{Ext}(\mathtt{IPC})$ *are mutually inverse complete lattice isomorphisms.*

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Theorem (Dummett-Lemmon conjecture, Dummett and Lemmon 1959; Zakharyashchev 1991)

For every $\mathtt{L} \in \mathsf{Ext}(\mathtt{IPC})$*, we have that* $\mathtt{L}$ *is Kripke complete iff* $\tau\mathtt{L}$ *is.*

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   - More!

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Our approach

We develop a new, uniform method for studying modal companions and notions in the vicinity, based on **stable canonical rules** [Bezhanishvili et al., 2016a].

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Advantages of our method vs. Zakharyashchev and Jeřábek's: stable canonical rules use **filtration**, canonical formulae and rules use (a version of) **selective filtration**.

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Advantages of our method vs. Zakharyashchev and Jeřábek's: stable canonical rules use **filtration**, canonical formulae and rules use (a version of) **selective filtration**.

- Filtration is simpler.
- Filtration is more easily generalizable to alternative signatures.

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1. **Modal companions** of superintuitionistic logics and rule systems
2. **Tense companions** of bi-superintuitionistic logics and rule systems
3. The **Kuznetsov-Muravitsky isomorphism** between $\mathsf{NExt}(\mathtt{KM})$ and $\mathsf{NExt}(\mathtt{GL})$, and its generalization to rule systems

Main results

	Modal companions		Tense companions		Kuznetsov Muravitsky	
	Logics	Rule systems	Logics	Rule systems	Logics	Rule systems
Characterization theorem	✓	✓	✓	✓	—	—
Blok-Esakia theorem	✓	✓	✓	✓	✓	✓
Dummett-Lemmon conjecture	✗	✓	✗	✓	—	—
Axiomatic characterization of ρ, τ, σ via scr	✗	✓	✗	✓	✗	✓(σ, ρ) only

Table: ✓: proved and known, ✓: proved and new, —: not applicable, ✗: not proved

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Uniform approach: all main results are proved using essentially the same techniques, with minor adaptations to fit signature

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Stable canonical rules

Si stable canonical rules

Let $\mathfrak{F}$ be a finite Esakia space, $\mathfrak{D} \subseteq \wp(F)$

$$(\mathfrak{F}, \mathfrak{D}) \qquad \mapsto \qquad \eta(\mathfrak{F}, \mathfrak{D})$$

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Modal stable canonical rules

Let $\mathfrak{F}$ be a finite $\mathtt{S4}$ modal space, $\mathfrak{D} \subseteq \wp(F)$

$$(\mathfrak{F}, \mathfrak{D}) \qquad \mapsto \qquad \mu(\mathfrak{F}, \mathfrak{D})$$

Refutation conditions

Proposition

For every Esakia space $\mathfrak{X}$ we have $\mathfrak{X} \nvDash \eta(\mathfrak{F}, \mathfrak{D})$ iff there is a continuous surjection $f : \mathfrak{X} \to \mathfrak{F}$ such that

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$$\uparrow f(x) \cap \mathfrak{d} \neq \varnothing \Rightarrow f[\uparrow x] \cap \mathfrak{d} \neq \varnothing.$$

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Proposition

For every modal space $\mathfrak{X}$ we have $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$ iff there is a continuous surjection $f : \mathfrak{X} \to \mathfrak{F}$ such that

- f is **stable**: Rxy implies $Rf(x)f(y)$;
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$$R[f(x)] \cap \mathfrak{d} \neq \varnothing \Rightarrow f[R[x]] \cap \mathfrak{d} \neq \varnothing.$$

Rewriting

Bezhanishvili et al. [2016a,b] prove the following results.

Theorem (Rewriting theorem (si))

Every si rule is equivalent to a conjunction of finitely many si stable canonical rules.

Theorem (Rewriting theorem (modal))

Every modal rule is equivalent, over $\mathtt{S4}$ *to finitely many modal stable canonical rules of the form* $\mu(\mathfrak{F}, \mathfrak{D})$, *for* $\mathfrak{F}$ *a finite* $\mathtt{S4}$ *frame*

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Proof sketch.

Use filtration to construct finite countermodels, then encode the latter into stable canonical rules. $\square$

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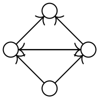

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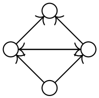

$\Rightarrow$

Both operations have algebraic duals $\sigma : \mathsf{HA} \to \mathsf{Grz}$, $\rho : \mathsf{S4} \to \mathsf{HA}$

Modal companion maps

Extend σ, ρ to class operators

$$\begin{aligned} \sigma &: \mathsf{Uni(HA)} \to \mathsf{Uni(Grz)} \\ \mathcal{U} &\mapsto \mathsf{Uni}\{\sigma\mathfrak{H} : \mathfrak{H} \in \mathcal{U}\} \end{aligned}$$

$$\begin{aligned} \rho &: \mathsf{Uni(S4)} \to \mathsf{Uni(HA)} \\ \mathcal{U} &\mapsto \{\rho\mathfrak{A} : \mathfrak{A} \in \mathcal{U}\} \end{aligned}$$

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Definition

Let $\mathtt{L} \in \mathsf{Ext}(\mathtt{IPC_R})$ be a si-rule system and $\mathtt{M} \in \mathsf{NExt}(\mathtt{S4_R})$ a modal rule system. We say that $\mathtt{M}$ is a *modal companion* of $\mathtt{L}$ (or that $\mathtt{L}$ is the si fragment of $\mathtt{M}$) whenever $\Gamma/\Delta \in \mathtt{L}$ iff $T(\Gamma/\Delta) \in \mathtt{M}$.

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$$\tau : \mathsf{Ext}(\mathtt{IPC_R}) \to \mathsf{NExt}(\mathtt{S4_R})$$
$$\mathtt{L} \mapsto \mathtt{S4_R} \oplus \{T(\Gamma/\Delta) : \Gamma/\Delta \in \mathtt{L}\}$$

$$\sigma : \mathsf{Ext}(\mathtt{IPC_R}) \to \mathsf{NExt}(\mathtt{S4_R})$$
$$\mathtt{L} \mapsto \mathtt{Grz_R} \oplus \tau\mathtt{L}$$

$$\rho : \mathsf{NExt}(\mathtt{S4_R}) \to \mathsf{Ext}(\mathtt{IPC_R})$$
$$\mathtt{M} \mapsto \{\Gamma/\Delta : T(\Gamma/\Delta) \in \mathtt{M}\}$$

Our Method

Basic idea

Use geometric refutation conditions for stable canonical rules to translate the target problems into order-topological problems.

The Blok-Esakia theorem for rule systems

Theorem

The mappings $\sigma : \mathsf{Ext}(\mathtt{IPC}_\mathtt{R}) \to \mathsf{NExt}(\mathtt{Grz}_\mathtt{R})$ *and* $\rho : \mathsf{NExt}(\mathtt{Grz}_\mathtt{R}) \to \mathsf{Ext}(\mathtt{IPC}_\mathtt{R})$ *are complete lattice isomorphisms and mutual inverses.*

Proof strategy

The tricky part consists in showing that for all $\mathtt{M} \in \mathsf{NExt}(\mathtt{Grz}_{\mathtt{R}})$ we have $\sigma\rho\mathtt{M} = \mathtt{M}$.

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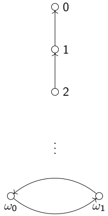

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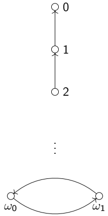

Main lemma

For all $\mathtt{Grz}$-spaces $\mathfrak{X}$, if $\mathfrak{X} \not\models \Gamma/\Delta$ then $\sigma\rho\mathfrak{X} \not\models \Gamma/\Delta$.

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- Transform f into a continuous stable surjection $g : \sigma\rho\mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$
- Ingredients: separation properties of Stone spaces, order-theoretic properties of `Grz`-spaces

The Dummett-Lemmon conjecture

Theorem (Dummett-Lemmon conjecture for si-rule systems)

For every si-rule system $\mathtt{L} \in \mathsf{Ext}(\mathtt{IPC}_\mathtt{R})$, *we have that* $\mathtt{L}$ *is Kripke complete iff* $\tau\mathtt{L}$ *is.*

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- Use Kripke completeness of $\mathtt{L}$ to get a Kripke frame $\mathfrak{X}$ validating $\mathtt{L}$ and a stable map $f : \mathfrak{X} \to \rho\mathfrak{F}$ satisfying the BDC for $\rho\mathfrak{D}$

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- Wlog, $\Gamma/\Delta = \mu(\mathfrak{F}, \mathfrak{D})$
- $\mu(\mathfrak{F}, \mathfrak{D}) \notin \tau\mathtt{L}$ implies $\eta(\rho\mathfrak{F}, \rho\mathfrak{D}) \notin \mathtt{L}$, where

$$\rho\mathfrak{D} = \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}\}$$

- Use Kripke completeness of $\mathtt{L}$ to get a Kripke frame $\mathfrak{X}$ validating $\mathtt{L}$ and a stable map $f : \mathfrak{X} \to \rho\mathfrak{F}$ satisfying the BDC for $\rho\mathfrak{D}$
- Expand $\mathfrak{X}$ into a Kripke frame $\mathfrak{Y}$ for $\tau\mathtt{L}$ by adding clusters, use f to define a map $g : \mathfrak{Y} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$

Overview

Further Work

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Thank You!

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