An algebraic theory of clones

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Clones

- A clone is a set of finitary operations on a set A containing all projections and closed under (many-sorted) composition

- *Universal algebra*: The set of all term operations of an algebra always forms a clone and in fact every clone is of this form.

- *First-order structures*: The *polymorphism clone* of a first-order structure $\mathbf{A}$, consisting of all finitary functions $f : A^n \to A$ which preserve the structure, carries information about the structure that induces it.

- *Theoretical computer science*: Many computational problems can be phrased as constraint satisfaction problems (CSPs).
  - CSP$(\mathbf{A})$ = the computational problem of deciding whether a conjunction of atomic formulas is satisfiable in a structure $\mathbf{A}$.
  - Jeavons has shown that, for a finite structure $\mathbf{A}$, the complexity of CSP$(\mathbf{A})$ is completely determined by the polymorphism clone of $\mathbf{A}$.

Clones into algebras

- Clones are many-sorted algebras: if $f : A^n \to A$ and $g_i : A^k \to A$, then $f(g_1, \ldots, g_n) : A^k \to A$ is defined as follows:

$$f(g_1, \ldots, g_n)(a_1, \ldots, a_k) = f(g_1(a_1, \ldots, a_k), \ldots, g_n(a_1, \ldots, a_k)).$$

- Abstract Clones equationally axiomatise clones as many-sorted algebras through a family of many-sorted composition operators C_k^n:

$$C_k^n(f, g_1, \ldots, g_n) = f(g_1, \ldots, g_n)$$

  and projections π_k^n $(1 \leqslant k \leqslant n)$. Every abstract clone is isomorphic to a clone of finitary operations.

- We formalise clones as one-sorted algebras.

Clone algebras

- We give the same domain to all finitary operations

- A finitary operation $f : A^n \to A$ becomes an infinitary operation $f^\top : A^\omega \to A$ (called the top extension of f)

$$f^\top(s) = f(s_1, \dots, s_n), \text{ for every } s \in A^\omega$$

- The composition becomes an operator q_n of arity $n+1$ (for every $n \geqslant 0$):

$$q_n^{A^\omega}(\varphi, \psi_1, \dots, \psi_n)(s) = \varphi(\psi_1(s), \dots, \psi_n(s), s_{n+1}, \dots), \text{ for every } s \in A^\omega$$

for arbitrary $\varphi, \psi_i : A^\omega \to A$

- $q_n^{A^\omega}(f^\top, g_1^\top, \dots, g_n^\top)(s) = f(g_1, \dots, g_n)(s_1, \dots, s_k)$

The definition of clone algebras

Definition 1 *A clone algebra of type τ (CA_τ) is an algebra $\mathcal{C} = (C, q_n^{\mathcal{C}}, \mathsf{e}_i^{\mathcal{C}}, \sigma^{\mathcal{C}})_{n\geqslant 0, i\geqslant 1, \sigma\in\tau}$ satisfying the following conditions:*

(C0) $\sigma^{\mathcal{C}} \in C$ *for every operator* $\sigma \in \tau$;

(C1) $q_n(\mathsf{e}_i, x_1, \dots, x_n) = x_i \ (1 \leqslant i \leqslant n)$;

(C2) $q_n(\mathsf{e}_j, x_1, \dots, x_n) = \mathsf{e}_j \ (j > n)$;

(C3) $q_n(x, \mathsf{e}_1, \dots, \mathsf{e}_n) = x \ (n \geqslant 0)$;

(C4) $q_n(x, y_1, \dots, y_n) = q_k(x, y_1, \dots, y_n, \mathsf{e}_{n+1}, \dots, \mathsf{e}_k)$ *($k > n$)*;

(C5) $q_n(q_n(x, y_1, \dots, y_n), z_1, \dots, z_n) = q_n(x, q_n(y_1, z_1, \dots, z_n), \dots, q_n(y_n, z_1, \dots, z_n))$.

- If C is a clone, then the set $C^\top = \{f^\top : A^\omega \to A \mid f \in C\}$ of all its top extensions determines a clone algebra, called the top extension of C

- A free algebra over countable generators $v_1, \dots, v_n, \dots$ is a clone algebra:
  - $\mathsf{e}_i = v_i$
  - $q_n(a, b_1, \dots, b_n) = E(a)$, where E is the unique endomorphism of the free algebra mapping v_i into b_i $(i = 1, \dots, n)$
  - $\sigma^{\mathcal{C}}$ is the equivalence class of the term $\sigma(v_1, \dots, v_n)$.

Functional clone algebras

- The most natural CAs are algebras of functions called *functional clone algebras* (FCA).
- A FCA$_\tau$ $\mathcal{F}$ with value domain A is determined by a set F of infinitary operations $f : A^\omega \to A$, containing the projections, the basic infinitary operations $\sigma^\mathcal{F}$ ($\sigma \in \tau$) and closed under the finitary composition q_n ($s \in A^\omega$):
  - $\mathsf{e}_i^{A^\omega}(s) = s_i$
  - $q_n^{A^\omega}(\varphi, \psi_1, \dots, \psi_n)(s) = \varphi(\psi_1(s), \dots, \psi_n(s), s_{n+1}, s_{n+2}, \dots)$
- **Theorem**: Clones $\Longleftrightarrow$ Finite-dimensional clone algebras.
- **Representation Theorem**: CA$_\tau$ $= \mathbb{I}$ FCA$_\tau$. (Difficult proof)
- A solution to the lattice of equational theories problem by Birkhoff and Maltsev (see also Newrly 1993 and Nurakunov 2008):

  Theorem: A lattice L is isomorphic to a lattice of equational theories iff L is isomorphic to the congruence lattice Con$(\mathcal{C})$ of a finite-dimensional clone algebra $\mathcal{C}$.

Another representation of clone algebras

- Let $\mathcal{C} = (C, q_n^{\mathcal{C}}, \mathsf{e}_i^{\mathcal{C}}, \sigma^{\mathcal{C}})$ be a clone τ-algebra and let $\epsilon^{\mathcal{C}} = (\mathsf{e}_1^{\mathcal{C}}, \dots, \mathsf{e}_n^{\mathcal{C}}, \dots)$. We define

$$[\epsilon^{\mathcal{C}}]_\omega = \{s \in C^\omega : |\{i : s_i \neq \mathsf{e}_i^{\mathcal{C}}\}| < \omega\}.$$

- $\mathcal{C}$ is isomorphic to a clone algebra of functions $\varphi_a : [\epsilon^{\mathcal{C}}]_\omega \to C$ $(a \in C)$, where

$$\varphi_a(s) = q_n^{\mathcal{C}}(a, s_1, \dots, s_n), \quad \text{if } s = \epsilon^{\mathcal{C}}[s_1, \dots, s_n] \in [\epsilon^{\mathcal{C}}]_\omega.$$

Traces and revisiting FCAs

- Let A be a set. We define an equivalence relation $\equiv_\omega$ on the set A^ω:

  $$r \equiv_\omega s \text{ iff } |\{i : r_i \neq s_i\}| < \omega$$

  $[r]_\omega$ is the equivalence class of $r \in A^\omega$.

- A trace a on A a nonempty subset of A^ω closed under $\equiv_\omega$.

- A t-operation is a function $\varphi : \mathsf{a} \to A$, whose domain is a trace a on A.

- A FCA with value domain A and trace a on A is a clone algebra of t-operations from a into A

Universal clone algebra I

- The most part of clone algebras are not finite-dimensional (e.g. the FCA of all infinitary operations).

- What are the algebraic structures that correspond to clone algebras in full generality?

$$\begin{bmatrix} \text{FinDim CA} & \text{Clone Algebras} \\ \text{Algebras} & ? \\ \text{Clones} & ? \end{bmatrix}$$

- Algebras and clones are to Universal Algebra what t-algebras and clone algebras are to Universal Clone Algebra.

Universal Clone Algebra II

- A **t-algebra** of type τ and trace a is a tuple $\mathbf{A} = (A, \mathsf{a}, \sigma^{\mathbf{A}})_{\sigma\in\tau}$, where $\sigma^{\mathbf{A}} : \mathsf{a} \to A$ is a t-operation for every $\sigma \in \tau$.

- $$\begin{bmatrix} \text{FinDim CA} & \text{Clone Algebras} \\ \text{Algebras} & \text{t-Algebras} \\ \text{Clones} & \text{FCA}s \end{bmatrix}$$

- We have two algebraic levels.
  The lower degree of t-algebras and the higher degree of clone algebras.

- We move between these levels either *individually* or *collectively*.

Clone Algebras of Terms

- Let τ be a set of operator symbols. The set T_τ of the τ-*terms* is built up by induction as follows:

  1. $\mathsf{e}_1, \dots, \mathsf{e}_n, \dots$ are terms;

  2. If $t_1, \dots, t_n$ are terms and $\sigma \in \tau$, then $\sigma(t_1, \dots, t_n, \mathsf{e}_{n+1}, \mathsf{e}_{n+2}, \dots)$ is a term, for every $n \geqslant 0$.

- The clone τ-algebra of τ-terms $\mathcal{T}_\tau = (T_\tau, q_n^{\mathcal{T}}, \mathsf{e}_i^{\mathcal{T}}, \sigma^{\mathcal{T}})_{\sigma \in \tau}$ is initial in the class of clone τ-algebras.

Up from t-algebras to CAs

- The term clone τ-algebra $\mathbf{A}^{\uparrow}$ **over** a t-algebra $\mathbf{A} = (A, \mathsf{a}, \sigma^{\mathbf{A}})_{\sigma \in \tau}$ is the minimal FCA of trace a containing all the t-operations $\sigma^{\mathbf{A}}$ of $\mathbf{A}$.

- For every $s \in \mathsf{a}$,

$$t^{\mathbf{A}}(s) = \begin{cases} s_i & \text{if } t \equiv \mathsf{e}_i \\ \sigma^{\mathbf{A}}(t_1^{\mathbf{A}}(s), \dots, t_n^{\mathbf{A}}(s), s_{n+1}, \dots) & \text{if } t \equiv \sigma(t_1, \dots, t_n, \mathsf{e}_{n+1}, \dots) \end{cases}$$

Down from CAs to t-algebras

- Let $\mathcal{C} = (C, q_n^{\mathcal{C}}, \mathsf{e}_i^{\mathcal{C}}, \sigma^{\mathcal{C}})$ be a clone τ-algebra

- The t-algebra $\mathcal{C}^{\downarrow} = (C, [\epsilon^{\mathcal{C}}]_\omega, \sigma^{\mathcal{C}^{\downarrow}})_{\sigma\in\tau}$ **under** a clone τ-algebra $\mathcal{C}$ is defined as follows: $\sigma^{\mathcal{C}^{\downarrow}} : [\epsilon^{\mathcal{C}}]_\omega \to C$ and

$$\sigma^{\mathcal{C}^{\downarrow}}(s) = q_n^{\mathcal{C}}(\sigma^{\mathcal{C}}, s_1, \dots, s_n) \text{ if } s = \epsilon^{\mathcal{C}}[s_1, \dots, s_n] \in [\epsilon^{\mathcal{C}}]_\omega$$

- If $\mathcal{C}$ is generated by the constants e_i $(i \in \omega)$ and $\sigma^{\mathcal{C}}$ $(\sigma \in \tau)$ then $\mathcal{C}^{\downarrow\uparrow} = \mathcal{C}$

t-Varieties and Et-varieties

- Let $\mathbf{A} = (A, \mathsf{a}, \sigma^{\mathbf{A}})$ be a t-algebra. *The subalgebra* $\mathbf{A}_s = (A_s, \mathsf{a}_s, \sigma^{\mathbf{A}_s})$ *of* $\mathbf{A}$ *generated by* $s \in \mathsf{a}$ is defined as follows:
  - $A_s = \{t^{\mathbf{A}}(s) : t \in T_\tau\}$ and $\mathsf{a}_s = [s]_\omega^{A_s}$.
  - $\sigma^{\mathbf{A}_s} = (\sigma^{\mathbf{A}})_{|\mathsf{a}_s}$

- A class K of t-algebras of type τ
  - is closed under expansion ($K = \mathbb{E}_t K$) if $(\forall s \in \mathsf{a}.\ \mathbf{A}_s \in K) \Rightarrow \mathbf{A} \in K$.
  - is closed under full expansion ($K = \mathbb{F}_t K$) if

$$(\text{For every minimal trace } \mathsf{b} \subseteq \mathsf{a}.\ \mathbf{A}_{\restriction \mathsf{b}} \in K) \Rightarrow \mathbf{A} \in K.$$

- A class K of t-algebras of type τ
  - is a t-variety if it is closed under $\mathbb{H}_t$, $\mathbb{S}_t$ and $\mathbb{P}_t$.
  - is an Ft-variety if it is a t-variety closed under $\mathbb{F}_t$.
  - is an Et-variety if it is a t-variety closed under $\mathbb{E}_t$.

Down of a class of clone algebras

- Let H be a class of CA$_\tau$s.
  - $H^\blacktriangledown = \{\mathbf{A} :$ there exists a FCA$_\tau$ with value domain $\mathbf{A}$ belonging to $H\}$.
  - $H^\triangledown = \{\mathbf{A} :$ there exists a PFCA$_\tau$ with value domain $\mathbf{A}$ belonging to $H\}$

  We have $H^\triangledown \subseteq H^\blacktriangledown$.

Theorem: If H is a variety of CA$_\tau$s, then

- $H^\blacktriangledown$ is an Et-variety of t-algebras.
- $H^\triangledown$ is a Ft-variety of t-algebras.

Up of a class of t-algebras

- K is a class of t-algebras of type τ.
- $K^{\triangle} = \mathbb{I}\{\mathcal{F} : \ \mathcal{F}$ is a FCA$_\tau$ with value domain $\mathbf{A} \in K\}$
- **Theorem** If K is a t-variety, then $K^{\triangle}$ is a variety of clone τ-algebras.

Generalised Birkhoff

Theorem (Birkhoff's Theorem 1 for t-algebras) Let K be a class of t-algebras of type τ. Then the following conditions are equivalent:

1. K is an Et-variety.
2. $K = \mathsf{Mod}(\mathsf{Th}_K)$.
3. $K^\vartriangle$ is a variety of CA$_\tau$s and $K = K^{\vartriangle\blacktriangledown}$.

Theorem (Birkhoff's Theorem 2 for t-algebras) Let K be a class of t-algebras of type τ. Then the following conditions are equivalent:

(i) K is an Ft-variety.

(ii) $K = \mathsf{Mod}(\mathsf{Th}_{K,X})$ for an infinite set X.

(iii) $K^\vartriangle$ is a variety of clone τ-algebras and $K = K^{\vartriangle\triangledown}$.

Classical Birkhoff for Algebras

Theorem (HSP Birkhoff)

- Let $\rho = (\rho_n : n \geqslant 0)$ be a finitary type and G be a class of ρ-algebras.
- Let $\rho^\star = \bigcup_{n \geqslant 0} \rho_n$ be the set of operator symbols (without arity).
- Let $G^\star$ be the class of t-algebras of type $\rho^\star$ obtained by gluing together algebras in G.

Then the following conditions are equivalent:

(i) G is a variety of ρ-algebras

(ii) G is an equational class of ρ-algebras

(iii) $G^\star$ is an Et-variety of t-algebras

(iv) $(G^\star)^\vartriangle$ is a variety of clone $\rho^\star$-algebras and $G^\star = (G^\star)^{\vartriangle\blacktriangledown}$

(v) $G^\star$ is an Ft-variety of t-algebras

(vi) $(G^\star)^\vartriangle$ is a variety of clone $\rho^\star$-algebras and $G^\star = (G^\star)^{\vartriangle\triangledown}$;

(vii) $G^\star$ is a t-variety of t-algebras.