# Gödel-Mckinsey-Tarski translation for non-distributive logics

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# Motivation

Gödel-Mckinsey-Tarski translation gives translation of modal logic to the S4 modal logics.

**Theorem (GMT translation)**

*There exists a translation* $\tau : \mathcal{L}_{\mathrm{IPC}} \to \mathcal{L}_{\mathrm{S4}}$ *such that for any* $\varphi \in \mathcal{L}_{\mathrm{IPC}}$*,*

$$\mathrm{IPC} \models \varphi \quad \textit{iff} \quad \mathrm{S4} \models \varphi$$

$$\begin{tikzcd}
\textbf{S4 Kripke frames} \arrow[r, shift left] \arrow[d] & \textbf{Intuitionistic frames} \arrow[l, shift left] \arrow[d] \\
\textbf{S4 modal algebras} \arrow[r, dashed, shift left] & \textbf{Heyting algebras} \arrow[l, dashed, shift left]
\end{tikzcd}$$

# Motivation

The main idea behind GMT translation is to emulate intuitionistic logic inside the classical logic.

**Applications**

- Transfer theorems
- Blok-Esakia theorem

Theorem

*The lattice of superintuitionistic logics is isomorphic to the lattice of normal expansions of Grzegorczyk modal logic.*

Grz axiom - $\Box(\Box(p \to \Box p) \to p) \to p$

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Can we try to model non-distributive logic (basic lattice logic) inside the classical logic in a similar manner?

# Basic lattice logic

**Language:** $\mathcal{L} \ni \varphi ::= p \in Prop \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$

**Lattice Logic:** Set of $\mathcal{L}$-sequents $\varphi \vdash \psi$

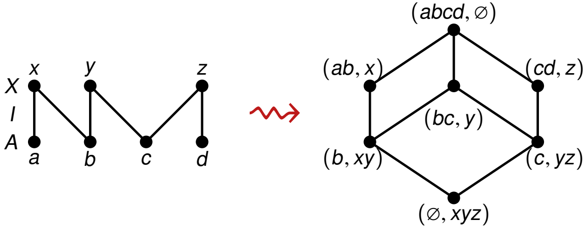
- containing:
  $p \vdash p \quad \bot \vdash p \quad p \vdash \top \quad p \vdash p \vee q \quad q \vdash p \vee q \quad p \wedge q \vdash p \quad p \wedge q \vdash q$
- closed under:
  $\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \quad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)} \quad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \quad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}$

**Semantics**

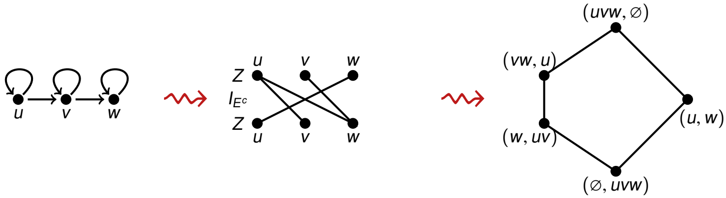
- Polarity semantics
- Graph-based semantics

# Relational semantics for LE-logics, via duality

## Polarities




## Reflexive graphs



# Graph-based semantics

A graph-based semantics is a frame $\mathbb{X} = (Z, E)$ such that $E$ is reflexive.

- The lattice corresponding to a graph-based frame $\mathbb{X}$ is given by $(Z, Z, E^c)^+$.
- For any lattice $\mathbb{L}$, the associated graph-based frame is $\mathbb{X} = (Z, E)$, where $Z = \{(F, I) \mid F \cap I = \emptyset\}$ and $(F_1, I_1)E(F_2, I_2)$ iff $F_1 \cap I_2 = \emptyset$.

# Graph-based models

A valuation for graph-based semantics is a map $v : \mathrm{Prop} \to \mathbb{F}^+$ such that $v(p) = ([\![p]\!], (\![p]\!))$.

A valuation provides information about both satisfaction and refutation of a variable.

The valuation extends naturally to the formulas.

- $V(\varphi \wedge \psi) = ([\![\varphi]\!] \cap [\![\psi]\!], ([\![\varphi]\!] \cap [\![\psi]\!])^{[1]})$.
- $V(\varphi \vee \psi) = (((\![\varphi]\!) \cap (\![\psi]\!))^{[0]}, (\![\varphi]\!) \cap (\![\psi]\!))$

Graph-based frames are just reflexive Kripke frames!

Can we define GMT like translation for non-distributive logic using reflexive Kripke frames?

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Graph-based frames are just reflexive Kripke frames!

Can we define GMT like translation for non-distributive logic using reflexive Kripke frames?

**Yes.**

# GMT translation for non-distributive (lattice) logic

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**Observations**

- Every graph-based frame valuation is a classical valuation.
- For every classical valuation $U$ the valuation $U^{[10]}$ is a graph-based frame valuation.

# GMT translation for non-distributive (lattice) logic

We want to build maps $\tau_1$ and $\tau_2$ corresponding to satisfaction and refutation.

**Semantic desiderata for translation**

For every classical assignment $U$ and graph-based frame assignment $V$,

1. $[\![\varphi]\!]_V = [\![\![\tau_1(\varphi)]\!]\!]_V$;
2. $[\![\![\tau_1(\varphi)]\!]\!]_U = [\![\varphi]\!]_{U^{[10]}}$.
3. $(\![\varphi]\!)_V = [\![\![\tau_2(\varphi)]\!]\!]^c_V$;
4. $[\![\![\tau_2(\varphi)]\!]\!]^c_U = (\![\varphi]\!)_{U^{[01]}}$ .

# GMT translation for non-distributive (lattice) logic

These conditions are satisfied by setting

$$\tau_1(\top) := \top \qquad \tau_1(\bot) := \rhd \blacktriangleright \bot \qquad \tau_1(p) := \rhd \blacktriangleright p,$$

and

$$\tau_2(\top) := \neg \blacktriangleright \top \qquad \tau_2(\bot) := \bot \qquad \tau_2(p) := \neg \blacktriangleright p, \ .$$

# GMT translation for non-distributive (lattice) logic

## Extending to meets and joins

$$[\![\tau_1(\varphi \vee \psi)]\!]_V = [\![\varphi \vee \psi]\!]_V = ([\![\tau_1(\varphi)]\!]_V^{[1]} \cap [\![\tau_1(\psi)]\!]_V^{[1]})^{[0]}.$$

$$[\![\tau_1(\varphi \wedge \psi)]\!]_V = [\![\varphi]\!]_V \cap [\![\psi]\!]_V = [\![\tau_1(\varphi)]\!]_V \cap [\![\tau_1(\psi)]\!]_V.$$

Dually,

$$[\![\tau_2(\varphi \vee \psi)]\!]_V^c = (\![\varphi \vee \psi]\!)_V = [\![\tau_2(\varphi)]\!]_V^c \cap [\![\tau_2(\psi)]\!]_V^c.$$

$$[\![\tau_2(\varphi \wedge \psi)]\!]_V^c = (\![\varphi \wedge \psi]\!)_V = (([\![\tau_2(\varphi)]\!]_V^c)^{[0]} \cap ([\![\tau_2(\psi)]\!]_V^c)^{[0]})^{[1]}.$$

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Dually,

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These conditions are satisfied by letting

$$\tau_1(\varphi \wedge \psi) := \tau_1(\varphi) \wedge \tau(\psi) \qquad \tau_1(\varphi \vee \psi) := \rhd(\blacktriangleright\tau_1(\varphi) \wedge \blacktriangleright\tau_1(\psi)),$$

and

$$\tau_2(\varphi \vee \psi) := \tau_2(\varphi) \vee \tau_2(\psi) \qquad \tau_2(\varphi \wedge \psi) := \neg \blacktriangleright (\rhd\neg\tau_2(\varphi) \wedge \rhd\neg\tau_2(\psi)).$$

# GMT translation for non-distributive (lattice) logic

Summing up, GMT translations $\tau_1, \tau_2 \colon \mathcal{L}_{\mathrm{LL}} \to \mathcal{L}_{\mathrm{T}}$ by the following recursion:

$$
\begin{array}{rcl}
\tau_1(p) & = & \rhd\blacktriangleright p \\
\tau_1(\bot) & = & \rhd\blacktriangleright\bot \\
\tau_1(\top) & = & \top \\
\tau_1(\varphi \wedge \psi) & = & \tau_1(\varphi) \wedge \tau_1(\psi) \\
\tau_1(\varphi \vee \psi) & = & \rhd(\blacktriangleright\tau_1(\varphi) \wedge \blacktriangleright\tau_1(\psi))
\end{array}
\qquad
\begin{array}{rcl}
\tau_2(p) & = & \neg\blacktriangleright p \\
\tau_2(\bot) & = & \bot \\
\tau_2(\top) & = & \neg \blacktriangleright \top \\
\tau_2(\varphi \wedge \psi) & = & \neg\blacktriangleright(\rhd\neg\tau_2(\varphi) \wedge \rhd\neg\tau_2(\psi)) \\
\tau_2(\varphi \vee \psi) & = & \tau_2(\varphi) \vee \tau_2(\psi) \,.
\end{array}
$$

**Theorem (GMT translation for lattice logic)**

*For every $\mathcal{L}_{\mathrm{LL}}$-formula $\varphi$, and every reflexive graph $\mathbb{X} = (Z, E)$,*

$$
\begin{array}{lcl}
\mathbb{X} \Vdash \varphi & \textit{iff} & \mathbb{X} \Vdash^* \tau_1(\varphi), \\
\mathbb{X} \succ \varphi & \textit{iff} & \mathbb{X} \not\Vdash^* \tau_2(\varphi).
\end{array}
$$

# Algebraic side of translation

We can now translate non-distributive modal logic into tense modal logic on reflexive frames.

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What about algebraic side?

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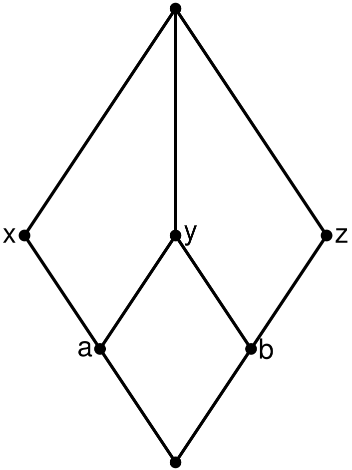
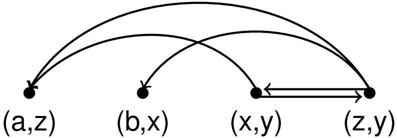
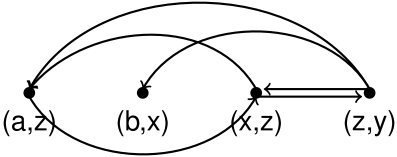
It's more **complicated**.

# Algebraic side of translation



# Algebraic side of translation

Clearly these frames do not give BAO belonging to the same variety.
The **Booleanization** of a lattice is not clear.

# Algebraic side of translation

For any lattice $\mathbb{L}$, let $\Phi(\mathbb{L})$ denote the set of tense modal algebras corresponding to it.

- $\Phi$ commutes with taking products.
- $\Phi$ does not commute with taking homomorphic images.
- $\Phi$ does not commute with taking subalgebras.

Can we still work out some transfer theorems?

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**We don't know.**

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# Conclusions and future directions

**Conclusions**

- Lattice logic can be translated into tense modal logic via GMT like translation.
- Translation has different satisfaction and refutation part.
- Algebraic side of translation is more complicated than in the case of Heyting algebras.

**Future directions**

- Restricting to special classes of lattices or graphs.
- Expanding the signature.
- Transfer theorems.