General and standard modal fuzzy logics

Amanda Vidal

Artificial Inteligence Research Institute IIIA - CSIC

TACL 2022, Coimbra, 23 June

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 101027914.

Table of contents

Table of Contents

Classical modal logics

- Modal logics: expand CL with non “truth-functional” operators
- K models naturally notions like ”possibly/necessarily”, ”sometimes/always”, and many other modal operators/logics are considered in the literature (deontic/temporal/conditional...)
- One of the first, best known, more studied, and applied non-classical logics.

(partially) why? offer a much higher expressive power than CPL and (generally) much lower complexity than FOL (most well-known and used modal logics are decidable).

Many-valued logics

- Many-valued logics: valuate the formulas out of $\{0,1\}(\top,\bot)$ and enrich the set of operations, to richer algebraic structures than **2**.
- Huge family of logics (different classes of algebras for evaluation). Allow modeling vague/uncertain/incomplete knowledge and probabilistic notions
- Very developed general theory (via algebraic logic and development in AAL)
- We will focus here in the three main so-called fuzzy logics: Gödel, Łukasiewicz, and Product, arising from continuous t-norms and their residua on [0,1].

(again) Richer logics, but for instance, above cases and Hàjek BL (which are infinitely-valued) still decidable.

Modal Many-valued logics

- Natural idea: expansion of MV logics with modal-like operators/interaction (or of modal-logics with wider algebraic evaluations/operations).
- Intuitionistic modal logics are particularly "nice": they naturally enjoy a relational semantics with a (rather) intuitive meaning.
- what about the rest? reasonable approach from the logical perspective: start from the (K) semantics and add the many-valuedness there $\longrightarrow$ valuation of Kripke models/frames over classes of algebras
  - Some modal MV logics have been axiomatised, but most have not. [Many usual intuitions fail, and usual constructions need to be adapted to get completeness -and in many cases, keep failing.] $\Rightarrow$ knowing the classes of modal algebras is dependent on finding these axiomatizations!
  - Decidability is hard to establish.
  - Relation to purely relational semantics is unknown.
  - Tools from classical modal logic like Sahlqvist theory have not been developed (wider set of operations + more specific semantics...)

Modal Many-valued logics - the semantical decission

- Even restricting to the interpretation of modal many-valued logic from before, the question of the definition of the logics is not fully determined:
  - At propositional level, the logics are characterized by their standard algebras (with universe $[0, 1]$).
  - We can consider the modal logics over those algebras, or over all algebras in the corresponding variety. Both would be modal fuzzy logics associated to that propositional logic.
- **In which cases do the previous logics coincide, or not?**

Table of Contents

What do we mean by "logic", and basic computability notions

Definition

A **Logic** $\vdash$ is a substitution invariant consequence relation on the algebra of the formulas. **[Not a set of formulas!]**

Definition

A set S is

- **Recursive/decidable**: there is an algorithm which takes "an input" and in a finite time determines whether it belongs to S or not.
- **Recursively enumerable (RE)** if there is an algorithm that enumerates the members of $S \equiv$ semidecidability.

"A logic $\vdash$ is RE" $\iff L = \{\langle \Gamma, \varphi \rangle : \Gamma \vdash \varphi, \Gamma \text{ finite}\}$ is RE. An axiomatization for $\vdash$ is a set $A \subseteq L$ s.t. L is the minimum logic containing A.

Craig's trick "adapted": RE logic $\iff$ recursively axiomatizable.

The non-modal part

Definition

A **BL-algebra** $\mathbf{A}$ is $\langle A, \odot, \rightarrow, \wedge, \vee, 0, 1\rangle$ such that

- $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice,
- $\langle A, \odot, 1\rangle$ is a commutative monoid
- $x \odot y \leq z \iff x \leq y \rightarrow z$ (residuation law)
- $x \wedge y = x \odot (x \rightarrow y)$,
- $(x \rightarrow y) \vee (y \rightarrow x) = 1$

$\Gamma \models_{\mathcal{C}} \varphi$ ($\Gamma \models_{\mathbf{A}} \varphi$) iff for any $\mathbf{A} \in \mathcal{C}$ and any $h \in Hom(Fm, \mathbf{A})$, if $h(\Gamma) \subseteq \{1\}$ then $h(\varphi) = 1$.

Classes of BL-algebras and logics in this talk

Gödel

$[0,1]_G$ $(\odot = \wedge)$, $\boldsymbol{G} := \mathbb{V}([0,1]_G)$. For **any** $\Gamma \subseteq Fm$,

$$\Gamma \vdash_G \varphi \iff \Gamma \models_{\boldsymbol{G}} \varphi \iff \Gamma \models_{[0,1]_G} \varphi$$

Łukasiewciz

$[0,1]_Ł$ $x \odot y = \max\{0, x+y-1\}$, $\boldsymbol{MV} := \mathbb{V}([0,1]_Ł)$. For **finite** $\Gamma \subseteq Fm$,

$$\Gamma \vdash_Ł \varphi \iff \Gamma \models_{\boldsymbol{MV}} \varphi \iff \Gamma \models_{[0,1]_Ł} \varphi$$

Product

$[0,1]_\Pi$ $\odot = \cdot$, $\boldsymbol{P} := \mathbb{V}([0,1]_\Pi)$. For **finite** $\Gamma \subseteq Fm$,

$$\Gamma \vdash_\Pi \varphi \iff \Gamma \models_{\boldsymbol{P}} \varphi \iff \Gamma \models_{[0,1]_\Pi} \varphi$$

(main "blocks" to build any other BL algebra, by using Ordinal Sum)

From modal (classical) logic...

- (minimal)Modal logic $\mathbf{K}$ = CPC +
  - K: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
  - $N_\Box$: from φ infer $\Box\varphi$ obs: over theorems $\Rightarrow$ local($\equiv$ set of theorems via D.T), over deductions $\Rightarrow$ global.
  - $\Diamond := \neg\Box\neg$

Definition

A *Kripke model* $\mathfrak{M}$ is a K. Frame $\mathfrak{F} = \langle W, R\rangle$ (W set, $R \subseteq W^2$) together with an evaluation $e \colon \mathcal{V} \to \mathcal{P}(W)$.

$$\mathfrak{M}, v \Vdash p \text{ iff } v \in e(p), \quad \mathfrak{M}, v \Vdash \neg\varphi \text{ iff } v \notin e(\varphi)$$

$$\mathfrak{M}, v \Vdash \varphi\{\wedge, \vee\}\psi \text{ iff } \mathfrak{M}, v \Vdash \varphi \text{ \{and, or\} } \mathfrak{M}, v \Vdash \psi$$

$$\mathfrak{M}, v \Vdash \Box\varphi \text{ iff for all } w \in W \text{ s.t. } R(v, w), \ \mathfrak{M}, w \Vdash \varphi$$

$$\mathfrak{M}, v \Vdash \Diamond\varphi \text{ iff there is } w \in W \text{ s.t. } R(v, w) \text{ and } \mathfrak{M}, w \Vdash \varphi$$

From modal (classical) logic...

- (minimal)Modal logic **K** $=$ CPC $+$
  - $K\colon \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
  - $N_\Box$: from φ infer $\Box\varphi$) obs: over theorems $\Rightarrow$ local($\equiv$ theorems via D.T), unrestricted (usual inference rule) *Rightarrow* global logic.
  - $\Diamond := \neg\Box\neg$

Definition

A *Kripke model* $\mathfrak{M}$ is a K. Frame $\mathfrak{F} = \langle W, R\rangle$ (W set, $R\colon W^2 \to \{0,1\}$) together with an evaluation $e\colon W \times \mathcal{V} \to \{0,1\}$.

$$e(v, \neg p) = \neg e(v,p), \qquad e(v, \varphi\{\wedge,\vee\}\psi) = e(v,\varphi)\{\wedge,\vee\}e(v,\psi)$$

$$e(v, \Box\varphi) = \begin{cases} 1 & \text{if for all } w \in W \text{ s.t. } R(v,w),\ e(u,\varphi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$e(v, \Diamond\varphi) = \begin{cases} 1 & \text{if there is } w \in W \text{ s.t. } R(v,w) \text{ and } e(w,\varphi) = 1 \\ 0 & \text{otherwise} \end{cases}$$

From modal (classical) logic...

- (minimal)Modal logic **K** = CPC +
  - $K: \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
  - $N_\Box$: from φ infer $\Box\varphi$) obs: over theorems/over deductions $\Rightarrow$ local($\equiv$ theorems via D.T)/global logic.
  - $\Diamond := \neg\Box\neg$

Definition

A *Kripke model* $\mathfrak{M}$ is a K. Frame $\mathfrak{F} = \langle W, R\rangle$ (W set, $R\colon W^2 \to \{0,1\}$) together with an evaluation $e\colon W \times \mathcal{V} \to \{0,1\}$.

$$e(v, \neg p) = \neg e(v, p), \qquad e(v, \varphi\{\wedge, \vee\}\psi) = e(v, \varphi)\{\wedge, \vee\}e(v, \psi)$$

$$e(v, \Box\varphi) = \bigwedge_{w \in W} \{Rvw \to e(w, \varphi)\}$$

$$e(v, \Diamond\varphi) = \bigvee_{w \in W} \{Rvw \wedge e(w, \varphi)\}$$

From modal (classical) logic(s)!...

There are in fact two logics.

- **(Local)**: $\Gamma \Vdash_K \varphi$ iff for all $\mathfrak{M}$ K-model and for all $w \in W$, $\mathfrak{M}, w \Vdash \Gamma \Rightarrow \mathfrak{M}, w \Vdash \varphi$ $e(w, [\Gamma]) \subseteq \{1\} \Rightarrow e(w, \varphi) = 1$
- **(Global)**: $\Gamma \Vdash^g_K \varphi$ iff for all $\mathfrak{M}$ K-model, $\mathfrak{M}, w \Vdash \Gamma$ for all $w \in W \Rightarrow \mathfrak{M}, w \Vdash \varphi$ for all $w \in W$ $e(w, [\Gamma]) \subseteq \{1\}$ for all $w \in W \Rightarrow e(u, \varphi) = 1$ for all $w \in W$

Completeness: $\Gamma \vdash_K \varphi \Leftrightarrow \Gamma \Vdash_K \varphi$ (resp. using K with $N_\Box$ over arbitrary deductions and $\Vdash^g_K$).

...to modal fuzzy logics

A BL algebra.

Definition

An **A**-*Kripke model* $\mathfrak{M}$ is a tripla $\langle W, R, e\rangle$ s.t. W is a set, $R\colon W^2 \to A)$ and $e\colon W \times V \to A$.

$$e(v, \varphi\{\wedge, \vee\}\psi) = e(v,\varphi)\{\wedge,\vee\}e(v,\psi)$$

$$e(v, \varphi \odot \psi) = e(v,\varphi) \odot e(v,\psi)$$

$$e(v, \varphi \to \psi) = e(v,\varphi) \to e(v,\psi)$$

$$e(v, \Box\varphi) = \bigwedge_{w\in W} \{R(v,w) \to e(w,\varphi)\}$$

$$e(v, \Diamond\varphi) = \bigvee_{w\in W} \{R(v,w) \odot e(w,\varphi)\}$$

safe whenever $e(u,\Box\varphi), e(u,\Diamond\varphi)$ are defined in every world.
crisp whenever $R\colon W^2 \to \{0,1\}$.

Modal logics over classes of BL-algebras

Let $\mathcal{A}$ be a class of BL algebras, and $\mathcal{K}$ be a class of **A**-Kripke models for $\mathbf{A} \in \mathcal{A}$.

- **(Local -over $\mathcal{K}$)**: $\Gamma \Vdash^{l}_{\mathcal{K}} \varphi$ iff for all $\mathfrak{M} \in \mathcal{K}$ and for all $w \in W$,

$$e(w, [\Gamma]) \subseteq \{1\} \Rightarrow e(w, \varphi) = 1$$

- **(Global -over $\mathcal{K}$)**: $\Gamma \Vdash^{g}_{\mathcal{K}} \varphi$ iff for all $\mathfrak{M} \in \mathcal{K}$,

$$e(w, [\Gamma]) \subseteq \{1\} \text{ for all } w \in W \Rightarrow e(u, \varphi) = 1 \text{ for all } w \in W$$

For $\mathcal{A}$ class of BL-algebras, $\Vdash^{*}_{\mathrm{M}\mathcal{A}}$ denotes the logics over all safe models over algebras in $\mathcal{A}$. $\Vdash^{*}_{\mathrm{K}\mathcal{A}}$ denotes the logics over all safe crisp models over algebras in $\mathcal{A}$.

$\Vdash^{*}_{\mathrm{M}\boldsymbol{G}}$, $\Vdash^{*}_{\mathrm{M}\boldsymbol{MV}}$, $\Vdash^{*}_{\mathrm{M}\boldsymbol{P}}$;
$\Vdash^{*}_{\mathrm{M}[0,1]_G}$, $\Vdash^{*}_{\mathrm{M}[0,1]_{Ł}}$, $\Vdash^{*}_{\mathrm{M}[0,1]_\Pi}$ (and some others might appear)

Relation to FO

These modal logics can be translated into fragments of the corresponding FO logics.

$$\langle x, v\rangle^\sharp := P_x(v) \qquad \langle \varphi \star \psi, v\rangle^\sharp := \langle \varphi, v\rangle^\sharp \star \langle \psi, v\rangle^\sharp$$

$$\langle \Box\varphi, v\rangle^\sharp := \forall w\ R(v,w) \to \langle \varphi, w\rangle^\sharp \qquad \langle \Diamond\varphi, v\rangle^\sharp := \exists w\ R(v,w) \odot \langle \varphi, w\rangle^\sharp$$

Observation

$$\Gamma \Vdash^{l}_{\mathsf{M}\mathcal{A}} \varphi \iff \langle \Gamma, c\rangle^\sharp \models_{\forall\mathcal{A}} \langle \varphi, c\rangle^\sharp \text{ for a constant } c,$$

$$\Gamma \Vdash^{g}_{\mathsf{M}\mathcal{A}} \varphi \iff \forall v\ \langle \Gamma, v\rangle^\sharp \models_{\forall\mathcal{A}} \forall v\ \langle \varphi, v\rangle^\sharp$$

$$\Gamma \Vdash^{l}_{\mathsf{K}\mathcal{A}} \varphi \iff \forall v, w\ R(v,w) \vee \neg R(v,w), \langle \Gamma, c\rangle^\sharp \models_{\forall\mathcal{A}} \langle \varphi, c\rangle^\sharp$$

$$\Gamma \Vdash^{g}_{\mathsf{K}\mathcal{A}} \varphi \iff \forall v, w\ R(v,w) \vee \neg R(v,w), \forall v\ \langle \Gamma, v\rangle^\sharp \models_{\forall\mathcal{A}} \forall v\ \langle \varphi, v\rangle^\sharp$$

Some relevant known results on these logics

- Modal standard Gödel (local and global) have been axiomatized ($M[0,1]_G$: Caicedo and Rodriguez, 2015), $K[0,1]_G$ Rodriguez and V., '20) -by a finite axiomatic system. Local is known to be decidable -via an alternative semantics (Caicedo et al. '17).
- Standard Łukasiewicz logics have been axiomatized using an infinitary axiomatic system (i.e., with an infinitary inference rule) (Hansoul and Teheux, '12)
- Similarly, for product with constants (V. '17).
- global standard Łukasiewicz and product are not recursively axiomatizable (V., '22).
- standard Łukasiewicz local is known to be decidable (V., '22), so recursively axiomatizable.

Table of Contents

Table of Contents

Gödel: Immediate (as in FO)

Observation

$$\Gamma \Vdash^{l}_{\mathrm{M}\boldsymbol{G}} \varphi \Longleftrightarrow \Gamma \Vdash^{l}_{\mathrm{M}[0,1]_G} \varphi$$

$$\Gamma \Vdash^{g}_{\mathrm{M}\boldsymbol{G}} \varphi \Longleftrightarrow \Gamma \Vdash^{g}_{\mathrm{M}[0,1]_G} \varphi$$

$$\Gamma \Vdash^{l}_{\mathrm{K}\boldsymbol{G}} \varphi \Longleftrightarrow \Gamma \Vdash^{l}_{\mathrm{K}[0,1]_G} \varphi$$

$$\Gamma \Vdash^{g}_{\mathrm{K}\boldsymbol{G}} \varphi \Longleftrightarrow \Gamma \Vdash^{g}_{\mathrm{K}[0,1]_G} \varphi$$

Thus, trivially also $Th(\mathrm{M}\boldsymbol{G}) = Th(\mathrm{M}[0,1]_G)$ and $Th(\mathrm{K}\boldsymbol{G}) = Th(\mathrm{K}[0,1]_G)$.

Easily seen by "traveling" to FO, where the stardard and the general logics coincide, since any two countable dense linearly ordered sets are isomorphic (as ordered sets).

Table of Contents

Łukasiewicz: Local case

Lemma

$$\Gamma \Vdash^{l}_{\mathsf{M}\boldsymbol{MV}} \varphi \iff \Gamma \Vdash^{l}_{\mathsf{M}[0,1]_{Ł}} \varphi$$
$$\Gamma \Vdash^{l}_{\mathsf{K}\boldsymbol{MV}} \varphi \iff \Gamma \Vdash^{l}_{\mathsf{K}[0,1]_{Ł}} \varphi$$

Thus, trivially also $Th(\mathsf{M}\boldsymbol{MV}) = Th(\mathsf{M}[0,1]_{Ł})$ and $Th(\mathsf{K}\boldsymbol{MV}) = Th([0,1]_{Ł})$.

- In [Hajek, '07] is proven that FO (general) Łukasiewicz is complete w.r.t. witnessed models.
- That can be inherited in $\Vdash^{l}_{\mathsf{M}\boldsymbol{MV}}$ and $\Vdash^{l}_{\mathsf{K}\boldsymbol{MV}}$. Since in the local deduction we also have completeness w.r.t. finite-depth models, we get completeness w.r.t. finite models with a particular structure determined by the formulas involved.
- We can encode all "modal information" propositionally with finitely many formulas, and use propositional completeness.

Łukasiewicz: crisp Global case

Lemma

$$\Gamma \Vdash^{g}_{\mathsf{K}\boldsymbol{MV}} \varphi \not\Longleftrightarrow \Gamma \Vdash^{g}_{\mathsf{K}[0,1]_{\L}} \varphi$$

- $\Vdash^{g}_{\mathsf{K}\boldsymbol{MV}}$ is R.E., because $\models_{\forall \boldsymbol{MV}}$ is R.E. and checking if the formulas are as in the translation to FO. is a decidable procedure.
- $\Gamma \Vdash^{g}_{\mathsf{K}[0,1]_{\L}} \varphi$ is not R.E. (V., '22)

Global case with an explicit counter-example

What about the non-crisp case?
In F.O., Hajek and Bou provided some rather complex examples encoding the theory of linear orders... In our case we can do it more directly. The difficult part is to find a *safe* countermodel.

Lemma

Let $\Phi = \{\Diamond 1, \Box s \leftrightarrow s \leftrightarrow \Diamond s \colon s \in \{y, p\}, p \to x, \Box x \leftrightarrow xy\}$. Then

$$1)\Phi \Vdash^{g}_{\mathrm{M}[0,1]_{Ł}} \neg p \vee y \qquad 2)\Phi \not\Vdash^{g}_{\mathrm{K}\boldsymbol{MV}} \neg p \vee y$$

- 1) can be checked "by hand":
- 2) $\langle \omega^{+}, \{n, n+1\}\rangle$ over Chang's MV algebra ($\Gamma(ZxZ, \langle 1, 0\rangle)$) with $e(n, p) = \langle 0, r\rangle, e(n, y) = \langle 1, -s\rangle, e(n, x) = \langle 1, -ns\rangle$.

Table of Contents

Product: Global case

Lemma

$$\Gamma \Vdash^{g}_{\mathsf{K}\boldsymbol{P}} \varphi \not\Longleftrightarrow \Gamma \Vdash^{g}_{\mathsf{K}[0,1]_{\Pi}} \varphi$$

The proof is the same as for Łukasiewicz.

Also, the previous counter-example works to prove that also

$$\Gamma \Vdash^{g}_{\mathsf{M}\boldsymbol{P}} \varphi \not\Longleftrightarrow \Gamma \Vdash^{g}_{\mathsf{M}[0,1]_{\Pi}} \varphi$$

by using $\langle \omega^{+}, \{n, n+1\} \rangle$ over the algebra $\mathfrak{B}(Z \times Z)$ with $e(n, p) = \langle -1, r \rangle, e(n, y) = \langle 0, -s \rangle, e(n, x) = \langle 0, -ns \rangle$.

Product: Local case

Lemma

$$\Gamma \Vdash^{l}_{\mathrm{M}\boldsymbol{P}} \varphi \Longleftrightarrow \Gamma \Vdash^{l}_{\mathrm{M}[0,1]_{\Pi}} \varphi$$

Thus, trivially also $Th(\mathrm{M}\boldsymbol{P}) = Th(\mathrm{M}[0,1]_{\Pi})$.

Follows from the same key idea used in the proof of decidability of $Th(\mathrm{M}[0,1]_{\Pi})$ (Cerami and Esteva '21).

It is crucial that Rvw takes values necessarly in $(0,1)$!

Product: Local case

Lemma

$$\Gamma \Vdash^{l}_{K\boldsymbol{P}} \varphi \Longleftrightarrow \Gamma \Vdash^{l}_{K[0,1]_{\Pi}} \varphi$$

Thus, trivially also $Th(K\boldsymbol{P}) = Th(K[0,1]_{\Pi})$.

Intuition:

1. identify a certain class of models wrt. to which the modal logic of the variety is complete, that satisfy some "good" properties that can be finitely expressed in the propositional language. This will allow us to move from $\Gamma \nVdash^{l}_{K\boldsymbol{P}} \varphi$ to $\Theta(\Gamma,\varphi) \nvdash_{\Pi} \phi(\Gamma,\varphi)$ for some -useful- finite $\Theta(\Gamma,\varphi), \phi(\Gamma,\varphi)$.
2. prove that the properties were "good enough", i.e., that from the $\Theta(\Gamma,\varphi) \nvdash_{\Pi} \phi(\Gamma,\varphi)$ we can build back an standard (crisp) Kripke model from which, indeed, $\Gamma \not\models^{l}_{K[0,1]_{\Pi}} \varphi$.

Some details of the proof

- FO Product logic (over $\boldsymbol{P}$) is complete w.r.t quasi-witnessed models (i.e., witnessed possibly except if $e(\Box\varphi) = 0$) [Hajek '98]
- More in particular [Laskowski-Malekpour, '07] proved it is complete w.r.t quasi-witnessed models over $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$, for $\mathbb{R}^{\mathbb{Q}}$ being the Lexicographic sum group: the ordered abelian group of functions $f\colon \mathbb{Q} \to \mathbb{R}$ whose support is well ordered (i.e., $\{q \in \mathbb{Q}\colon f(q) \neq 0\}$ is a well ordered subset of $\mathbb{Q}$). $+$ is defined component-wise and the ordering is lexicographic.

What can we say about unwitnessed formulas in $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-models?

Proposition

Let Ω be a finite closed set of formulas, and $\mathfrak{M}$ be a quasi-witnessed $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-Kripke model. Then, there is a model $\mathfrak{M}'$ extending $\mathfrak{M}$ such that for each $v \in W$ and $\psi \in \Omega$ it holds that $e'(v, \psi) = e(v, \psi)$, and such that, for each $v \in W'$ and each $\varphi \in \Omega$ unwitnessed in v there are two worlds $v_\varphi, \overline{v_\varphi}$ such that

1. Rvv_φ and $Rv\overline{v_\varphi}$,
2. For each formula $\delta \in \Omega$ there exists $a_{\delta, v_\varphi} \in \mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$ such that $e(\overline{v_\varphi}, \delta) = e(v_\varphi, \delta) + a_{\delta, v_\varphi}$, and
    - $\bot < a_{\varphi, v_\varphi} < \top$, and $a_{\delta, v_\varphi} = \top$ for each $\Box\delta \in \Omega$ with $e(v, \Box\delta) > \bot$,
    - $e(v_\varphi, \delta) \leq e(v_\varphi, \gamma)$ implies $a_{\delta, v_\varphi} \leq a_{\gamma, v_\varphi}$ and $a_{\delta, v_\varphi} = \bot$ if and only if $e(v_\varphi, \delta) = \bot$.

What can we say about unwitnessed formulas in $\mathfrak{B}(\mathbb{R}^{\mathbb{Q}})$-models?

$e(v, \varphi) = \bot$ unwit:

$\bigwedge e(v,\varphi) = \bot \Longrightarrow \exists q \in \mathbb{Q}$ and $u \in W$ with Rvu s.t.

1) $e(u,\chi)[p] = 0$ for all $p \leq q$, all $\chi \in \Sigma$ s.t. $e(v, \Box\chi) > 0$

2) $e(u,\varphi)[q] < 0$.

For $\left| \begin{array}{l} a \in \mathfrak{B}(\mathbb{R}^{\mathbb{Q}}) \\ q \in \mathbb{Q} \end{array} \right.$, let $a_{\leftarrow q} = \bot$ if $a = \bot$, and $a_{\leftarrow q}[p] = \begin{cases} 0 & \text{if } p > q \\ a[q] & \text{if } p \leq q \end{cases}$

Take the u above, and consider an additional $\bar{u}$ with

$$e(\bar{u}, p) = e(u,p)_{\leftarrow q}.$$

(This must be done with the generated submodels)

Syntactic translation of formulas

Let Υ be a finite set of (modal) formulas with maximum modal depth $n \geq 1$. For $0 \leq i \leq n$ let:

$$\Upsilon_0 := PropSFm(\Upsilon)$$
$$\Upsilon_{i+1} := \bigcup_{\heartsuit\psi \in \Upsilon_i} PropSFm(\psi)$$

We can use all sequences $\sigma = \langle \varphi_0, \ldots, \varphi_k \rangle$ for $\varphi_i \in \Upsilon_k$ **beginning with a modality** to encode the "witness" worlds from the previous model.

Further, to encode the identified pair of worlds associated to unwitnessed formulas, we consider also the necessary sequences of the form $\langle \varphi_1, \ldots, \overline{\varphi_k} \rangle$.

Call all these possible sequences Σ (and Σ_i the corresponding i-long sequences).

Syntactic translation of formulas

We will use the sequences Σ to generate a propositional language with variables $\mathcal{V}_\sigma$, $\heartsuit\varphi_\sigma$ and, for $\sigma \in \Sigma_i$ with last element $\overline{\varphi}$, new variables $\alpha_{\chi,\sigma}$ for $\chi \in \Upsilon_{i+1}$.

For each $\sigma \in \Sigma_i$ fix some set $uWit_\sigma \subseteq \Upsilon_i^\square$.

Definition

- $2V(\varphi_{\sigma\overline{\square\psi}}) := \varphi_{\sigma\overline{\square\psi}} \leftrightarrow \varphi_{\sigma\square\psi} \odot \alpha_{\varphi,\sigma\overline{\square\psi}}$,
- $Imp(\varphi_{\sigma\overline{\square\chi}}, \psi_{\sigma\overline{\square\chi}}) := \Delta(\varphi \to \psi)_{\sigma\square\chi} \to (\alpha_{\varphi,\sigma\overline{\square\chi}} \to \alpha_{\psi,\sigma\overline{\square\chi}})$,
- $Neg(\varphi_{\sigma\overline{\square\chi}}) := \neg\alpha_{\varphi,\sigma\overline{\square\chi}} \to \neg\varphi_{\sigma\square\chi}$,
- $WV(\Upsilon) := \bigwedge\{\neg\neg(\square\varphi)_\sigma \to \alpha_{\varphi,\sigma\overline{\square\chi}} \colon \alpha_{\varphi,\sigma\overline{\square\chi}} \in \mathcal{V}, \square\varphi \in \Upsilon_i\}$,
- $uWV(\Upsilon) := \bigvee\{\alpha_{\chi,\sigma\overline{\square\chi}} \colon \alpha_{\chi,\sigma\overline{\square\chi}} \in \mathcal{V}, \square\chi \in uWit_\sigma\}$,
- $W_\Diamond((\Diamond\psi)_\sigma) := ((\Diamond\psi)_\sigma \leftrightarrow (\psi)_{\sigma\Diamond\psi}) \wedge (\bigvee_{\sigma\chi\in\Sigma}(\psi)_{\sigma\chi} \to (\Diamond\psi)_\sigma)$,
- $W_\square((\square\psi)_\sigma) := ((\square\psi)_\sigma \leftrightarrow (\psi)_{\sigma\square\psi}) \wedge ((\square\psi)_\sigma \to \bigwedge_{\sigma\chi\in\Sigma}(\psi)_{\sigma\chi})$,
- $uW((\square\psi)_\sigma) := \neg(\square\psi)_\sigma$

Moving to propositional logic

Selecting only the sequences in Σ arising from the chosen $uWit_\sigma$ sets, and the previous definitions over the formulas of the corresponding level, we let

$$M(\Upsilon) := 2V(\Upsilon) \cup Imp(\Upsilon) \cup Neg(\Upsilon) \cup WV(\Upsilon) \cup W_\Diamond(\Upsilon) \cup W_\Box(\Upsilon) \cup uW(\Upsilon).$$

Theorem

Let $\Upsilon = \Gamma \cup \{\varphi\}$ be such that $\Gamma \not\models^{l}_{\boldsymbol{KP}} \varphi$. Then, for each sequence $\sigma \in \Sigma_i$ there exists a set $uWit_\sigma \subseteq \Upsilon_i^\Box$ such that

$$\Gamma_{\langle 0 \rangle}, M(\Upsilon) \nvdash_{\Pi_\Delta} \varphi_{\langle 0 \rangle} \vee uWV(\Upsilon)$$

and what does this propositional entailment "know"?

Proposition

Let Γ be a closed set of propositional formulas, and $h_1, h_2 \in Hom(Fm, [0,1]_\Pi)$ such that

1. For each formula $\varphi \in \Gamma$, there is some α_φ such that $h_2(\varphi) = h_1(\varphi) \cdot \alpha_\varphi$,
2. For each pair of formulas $\varphi, \psi \in \Gamma$ such that $h_1(\varphi) \leq h_1((\psi)$ it holds that $\alpha_\varphi \leq \alpha_\psi$,
3. $\alpha_\varphi = 0$ implies that $h_1(\varphi) = 0$.

Consider the family of homomorphisms h_k for $k \in \mathbb{N}$ where $h_k(x) = h(x) \cdot \alpha_x^k$ for each variable x in Γ.
Then, for each $\varphi \in \Gamma$, it holds that $h_k(\varphi) = h(\varphi) \cdot \alpha_\varphi^k$.

(C1) $\alpha_{\varphi \odot \psi} = \alpha_\varphi \cdot \alpha_\psi$ and (C2) $\alpha_{\varphi \to \psi} = \alpha_\varphi \to_{[0,1]_\Pi} \alpha_\psi$.

Back to an standard Kripke model

Lemma

Let $\Upsilon = \Gamma \cup \{\varphi\} \subset Fm$, and assume that for each sequence $\sigma \in \Sigma_i$ there exists a set $uWit_\sigma \subseteq \Upsilon^{\Box}_{k+1}$ such that

$$\Gamma_{\langle 0 \rangle}, M(\Upsilon) \not\Vdash_{\Pi_\Delta} \varphi_{\langle 0 \rangle} \vee uWV(\Upsilon)$$

Then, $\Gamma \not\Vdash^{l}_{K[0,1]_\Pi} \varphi$.

Bonus result:

Theorem

$\Vdash^{l}_{K[0,1]_\Pi}$ is decidable.

Muito obrigado!

(very short) Bibliography

Caicedo, X., Metcalfe, G., Rodríguez, R., and Rogger, J. (2017). Decidability of order-based modal logics. *Journal of Computer and System Sciences*, 88:53-74.

Cerami, M. and Esteva, F. (2021) On decidability of concept satisfiability in Description Logic with product semantics. *Fuzzy Sets and Systems*, (In press)

Hansoul, G. and Teheux, B. (2013). Extending Łukasiewicz logics with a modality: Algebraic approach to relational semantics. *Studia Logica*, 101(3):505–545.

M. C. Laskowski and S. Malekpour (2007) Provability in predicate product logic. *Archive for Mathematical Logic*, 46(5): 365–378.

Rodríguez, R. O. and Vidal, A. (2020). Axiomatization of Crisp Gödel Modal Logic. *Studia Logica*, 109: 367–395, 2020.

Vidal, A. (2017) On modal extensions of Product fuzzy logic *Journal of Logic and Computation, 27(1): 299–336*

Vidal, A. (2022) Undecidability and non-axiomatizability of modal many-valued logics. The Journal of Symbolic Logic, (In press)