Sahlqvist theory for deductive systems

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Let's start

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- Heyting algebras, i.e., structures $\boldsymbol{A} = \langle A; \wedge, \vee, \to, 0, 1 \rangle$ that comprise a bounded lattice $\langle A; \wedge, \vee, 0, 1 \rangle$ and satisfy

$$a \wedge b \leqslant c \Longleftrightarrow a \leqslant b \to c, \text{ for every } a, b, c \in A.$$

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In addition, every Heyting algebra $\boldsymbol{A}$ embeds into $\mathsf{Up}(\boldsymbol{A}_*)$.

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- a Sahlqvist antecedent (SA) if it is constructed from atoms, negative formulas, and 0 and 1 using only $\wedge$ and $\vee$;
- a Sahlqvist implication (SI) if it is positive, or of the form $\neg\varphi$ for a SA φ, or of the form $\varphi \to \psi$ for a SA φ and a positive ψ;
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Remark. Sahlqvist formulas are of the form $\varphi = \varphi_1 \vee \ldots \vee \varphi_n$. For example, $x \vee \neg x$ and $(x \to y) \vee (y \to x)$ are Sahlqvist.

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Definition

A Sahlqvist quasiequation is an expression of the form

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Remark

For every Heyting algebra $\boldsymbol{A}$ it holds

$$\boldsymbol{A} \vDash \Phi \quad \text{iff} \quad \boldsymbol{A} \vDash \varphi_1 \vee \ldots \vee \varphi_n.$$

We move to fragments

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Sahlqvist Canonicity for fragments of IPC with $\wedge$.

Let Φ be a Sahlqvist quasiequation in the language of a fragment L of IPC comprising $\wedge$. For every L-subreduct $\boldsymbol{A}$ of a Heyting algebra,

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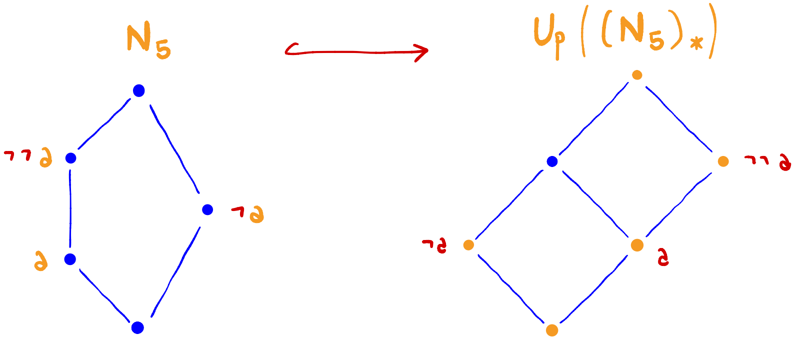

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Since $\mathrm{Up}(\boldsymbol{B}_*)$ validates Φ, so does $\mathrm{Up}(\boldsymbol{A}_*)$. $\square$

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Remark. The formula btw_n cannot be rendered as an equation!

A quick detour in algebraic logic

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  Most logics with a very weak implication $\to$ are protoalgebraic as witnessed by the set $\Delta = \{x \to y\}$.

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- The spectrum of an algebra $\boldsymbol{A}$ is the poset $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ of the meet irreducible deductive filters of $\vdash$ on $\boldsymbol{A}$.

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Intuitionistic logic has the IL witnessed by the set $\sim_n := \{\neg(x_1 \wedge \cdots \wedge x_n)\}$.

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- the proof by cases (PC) if there is $x \curlyvee y$ s.t.

$$\Gamma, \varphi \vdash \gamma \text{ and } \Gamma, \psi \vdash \gamma \text{ iff } \Gamma, \varphi \curlyvee \psi \vdash \gamma.$$

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- the deduction theorem (DT) if there is $x \Rightarrow y$ s.t.

$$\Gamma, \varphi \vdash \psi \text{ iff } \Gamma \vdash \varphi \Rightarrow \psi;$$

- the proof by cases (PC) if there is $x \curlyvee y$ s.t.

$$\Gamma, \varphi \vdash \gamma \text{ and } \Gamma, \psi \vdash \gamma \text{ iff } \Gamma, \varphi \curlyvee \psi \vdash \gamma.$$

Theorem (Blok & Pigozzi, Czelakowski & Dziobiak, Raftery)

A protoalgebraic logic $\vdash$ has the IL (resp. DT, PC) iff the semilattice $\mathrm{Fi}_{\vdash}^{\omega}(\boldsymbol{A})$ is pseudocomplemented (resp. implicative semilattice, distributive lattice) for every algebra $\boldsymbol{A}$.

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- $\mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$ for every algebra $\boldsymbol{A}$.

Proof sketch by example.

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- Finally, by Correspondence, $\mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$. □

Any applications?

- Recap on Sahlqvist theory (for intuitionistic logic);
- Sahlqvist theory for fragments of intuitionistic logic;
- Sahlqvist theory for protoalgebraic logics;
- Applications.

Examples. Let $\vdash$ be a protoalgebraic logic with the IL.

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Corollary (Lávička & Přenosil)

The logic $\vdash$ validates the following metarules for $n \in \mathbb{Z}^+$:

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Corollary (for $n = 1$, Lávička, M., Raftery)

The logic $\vdash$ validates the following metarules for $n \in \mathbb{Z}^+$:

$$\frac{\Gamma, \sim (\vec{\gamma}_1 \cup \cdots \cup \vec{\gamma}_{i-1} \cup \sim \vec{\gamma}_i) \rhd \psi \text{ for every } 1 \leqslant i \leqslant n+1}{\Gamma \rhd \psi}$$

iff it has bounded top width n: the principal upsets in $\text{Spec}_{\vdash}(\boldsymbol{A})$ have at most n maximal elements, for every $\boldsymbol{A}$.

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Let L be a fragment of IPC comprising $\to$. For every L-subreduct $\boldsymbol{A}$ of a Heyting algebra,

$$\text{if } \boldsymbol{A} \vDash \Phi^*, \text{ then } \mathrm{Up}(\mathrm{Spec}_{\mathrm{L}}(\boldsymbol{A})) \vDash \Phi^*,$$

where $\mathrm{Spec}_{\mathrm{L}}(\boldsymbol{A})$ is the poset of meet irr. implicative filters of $\boldsymbol{A}$.

One last example.

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Correspondence for intuitionistic linear logics.

Let $\Phi = \varphi_1 \wedge y \leqslant z \,\&\ldots\&\, \varphi_m \wedge y \leqslant z \Longrightarrow y \leqslant z$ be a Sahlqvist quasiequation compatible with an axiomatic extension $\vdash$ of ILL. The theorems of $\vdash$ include the formula $(1 \wedge \boldsymbol{\varphi}^1_{\mathbf{1}}) \vee \cdots \vee (1 \wedge \boldsymbol{\varphi}^1_{\boldsymbol{m}})$ iff $\mathsf{Spec}(\boldsymbol{A}) \vDash \mathsf{tr}(\Phi)$, for every algebra $\boldsymbol{A} \in \mathsf{K}_{\vdash}$.

Thank you very much for your attention!