Non-distributive logics as evidential logics

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Motivation

- mathematical theory of LE-logics (LE: lattice expansions)
- algebraic and Kripke-style semantics
- generalized Sahlqvist theory
- algebraic proof theory (semantic cut elimination, FMP)
- Goldblatt-Thomason theorem
- unified inverse correspondence
- many-valued semantics

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Can we make intuitive sense of LE-logics?

Basic lattice logic & main ideas

Language: $\mathcal{L} \ni \varphi ::= p \in Prop \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$

Lattice Logic: Set of $\mathcal{L}$-sequents $\varphi \vdash \psi$

- containing:

  $p \vdash p \quad \bot \vdash p \quad p \vdash \top \quad p \vdash p \vee q \quad q \vdash p \vee q \quad p \wedge q \vdash p \quad p \wedge q \vdash q$

- closed under:

  $\dfrac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \quad \dfrac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)} \quad \dfrac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \quad \dfrac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}$

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Challenge: Interpreting $\vee$ as ‘or’ and $\wedge$ as ‘and’ does not work, since ‘and’ and ‘or’ distribute over each other, while $\wedge$ and $\vee$ don’t.

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Proposal (Graph-based semantics): Interpreting $\varphi \in \mathcal{L}$ as sentences under **other circumstances** (e.g. filtered through informational entropy)

Non-distributive logics, aka normal LE-logics

LE: Lattice Expansions: $\mathbb{A} = (\mathbb{L}, \mathcal{F}^{\mathbb{A}}, \mathcal{G}^{\mathbb{A}})$
lattice signature + operations of any finite arity.
Additional operations partitioned in families $f \in \mathcal{F}$ and $g \in \mathcal{G}$.
Normality: In each coordinate,

- f-type operations *preserve* finite **joins** in positive coordinates and *reverse* finite **meets** in negative coordinates;
- g-type operations *preserve* finite **meets** in positive coordinates and *reverse* finite **joins** in negative coordinates.

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Examples

- Distributive Modal Logic: $\mathcal{F} := \{\Diamond, \lhd\}$ and $\mathcal{G} := \{\Box, \rhd\}$
- Bi-intuitionistic modal logic: $\mathcal{F} := \{\Diamond, \succ\!\!-\}$ and $\mathcal{G} := \{\Box, \rightarrow\}$
- Full Lambek calculus: $\mathcal{F} := \{\circ\}$ and $\mathcal{G} := \{/, \backslash\}$
- Lambek-Grishin calculus: $\mathcal{F} := \{\circ, /_{\oplus}, \backslash_{\oplus}\}$ and $\mathcal{G} := \{\oplus, /_{\circ}, \backslash_{\circ}\}$
- $\ldots$

Modelling informational entropy

Informational entropy: an inherent boundary to knowability, due e.g. to perceptual, theoretical, evidential or linguistic limits.

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- limit **incorporated into meaning of connectives**
  (compare with intuitionistic interpretation of $\rightarrow$)

Informational Entropy as “intuitionism on speed”

Comparison with relational semantics of **intuitionistic logic**:

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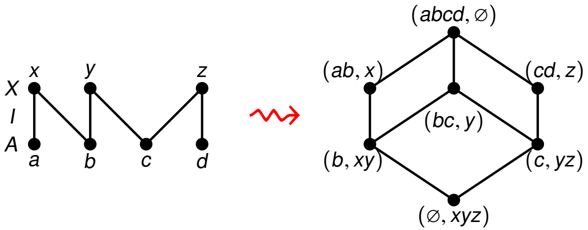
truth $\rightsquigarrow$ provability $\rightsquigarrow$ evidential reasoning

Relational semantics for LE-logics, via duality

Polarities (Birkhoff)



Relational semantics for LE-logics, via duality

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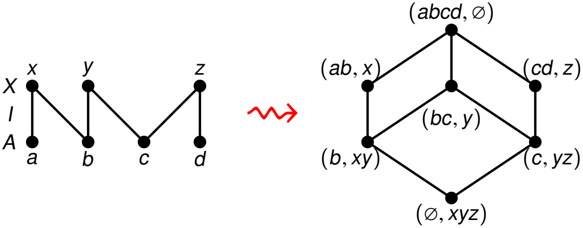
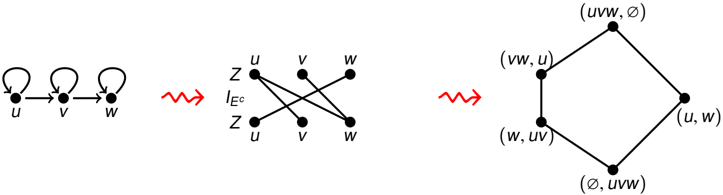
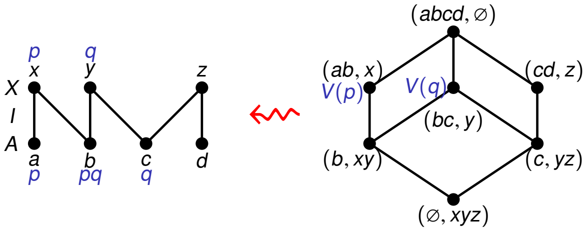


Reflexive graphs (Ploščica, 1995)

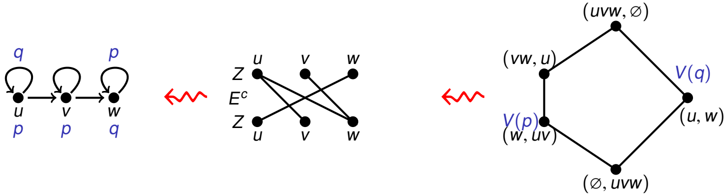


Compositional semantics for basic lattice logic

Polarities (Gehrke)



Reflexive graphs (Conradie & Craig)



$z \Vdash \varphi$ iff $z \in [\![\varphi]\!]$ $\qquad$ $z \succ \varphi$ iff $z \in (\![\varphi]\!)$

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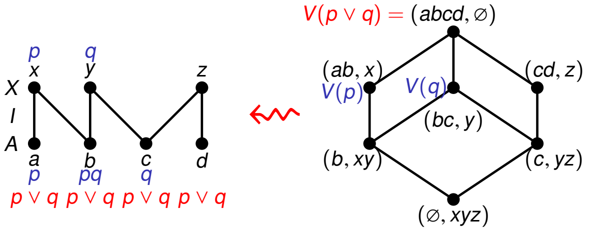

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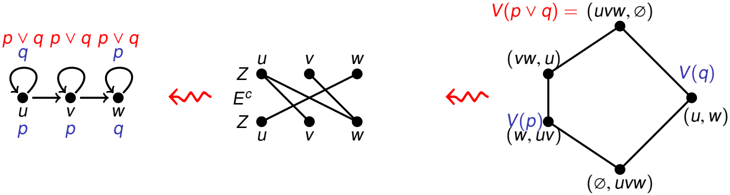
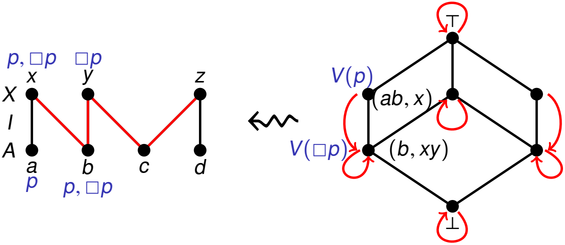


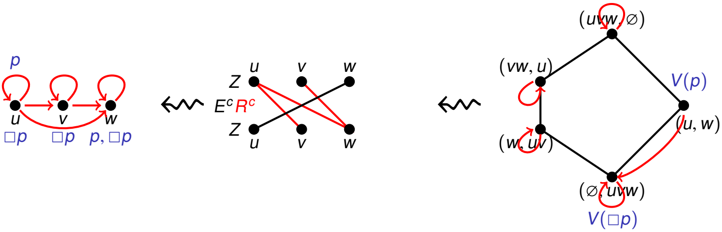
$z \Vdash \varphi$ iff $z \in [\![\varphi]\!]$ $\qquad$ $z \succ \varphi$ iff $z \in (\![\varphi]\!)$

Compositional semantics, expanded signature

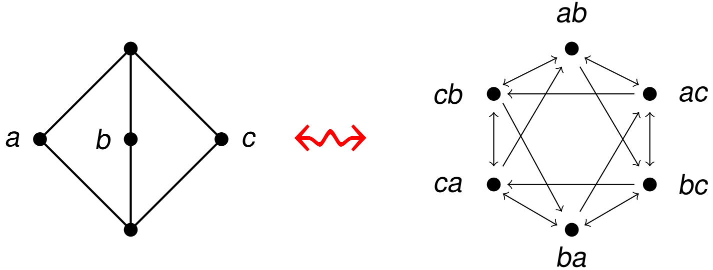
Polarity-based frames (Gehrke)



Graph-based frames (Conradie & Craig)



Graph-based semantics of LE-logics



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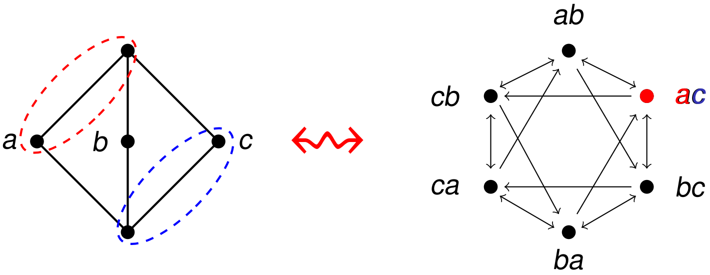
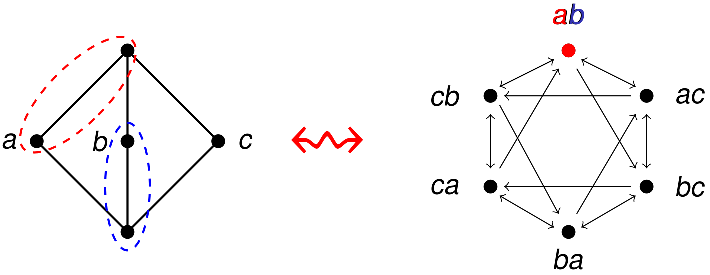


Representation. States: maximally disjoint filter-ideal pairs (F, I);

$$(F, I)\ E\ (F', I') \quad \text{iff} \quad F \cap I' = \varnothing$$

Graph-based semantics of LE-logics



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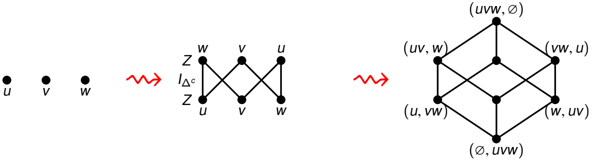
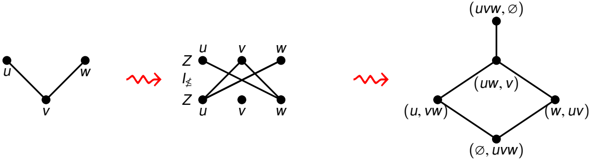
Reflexive graphs as generalized intuitionistic frames

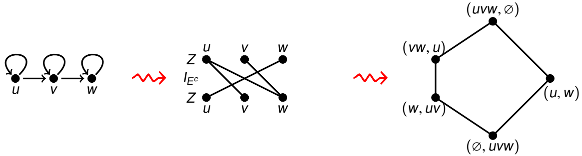
Sets



Posets



Reflexive graphs



Summary of the definitions

$$
\begin{array}{lcc}
z \Vdash \varphi \wedge \psi & \text{iff} & z \Vdash \varphi \text{ and } z \Vdash \psi \\
z \succ \varphi \wedge \psi & \text{iff} & \text{for all } z', \text{ if } zEz' \text{ then } z' \nVdash \varphi \wedge \psi \\
\\
z \Vdash \varphi \vee \psi & \text{iff} & \text{for all } z', \text{ if } zEz' \text{ then } z' \not\succ \varphi \vee \psi \\
z \succ \varphi \vee \psi & \text{iff} & z \succ \varphi \text{ and } z \succ \psi \\
\\
z \Vdash \Box\varphi & \text{iff} & \text{for all } z', \text{ if } zR_{\Box}z' \text{ then } z' \not\succ \varphi \\
z \succ \Box\varphi & \text{iff} & \text{for all } z', \text{ if } z'Ez \text{ then } z' \nVdash \varphi \\
\\
z \Vdash \Diamond\varphi & \text{iff} & \text{for all } z', \text{ if } zEz' \text{ then } z' \not\succ \varphi \\
z \succ \Diamond\varphi & \text{iff} & \text{for all } z', \text{ if } zR_{\Diamond}z' \text{ then } z' \nVdash \varphi \\
 & \ldots &
\end{array}
$$

Evidential logic as hyper-constructivism

If $z \Vdash \varphi$ is interpreted as

'In z we have evidence to **accept** φ',

and $z \succ \varphi$ is interpreted as

'In z we have evidence to **refute** φ',

then φ denote propositions in a **hyper-constructivist** context:

$z \nVdash \varphi$ does **not** imply $z \succ \varphi$

meta-linguistic failure of 'excluded middle'.

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Reflexivity as E-reflexivity

$$
\begin{array}{ll}
 & \forall p[\Box p \leq p] \\
\text{iff} & \forall \mathbf{j}[\mathbf{j} \leq \blacklozenge \mathbf{j}] \\
\text{iff} & \forall z[z^{[10]} \subseteq R^{[0]}[z^{[1]}]] \\
\text{iff} & E \subseteq R
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Transitivity as E-transitivity

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\text{iff} & \forall \mathbf{j}[\blacklozenge\blacklozenge \mathbf{j} \leq \blacklozenge \mathbf{j}] \\
\text{iff} & \forall z[R^{[0]}[(R^{[0]}[z^{[01]}])^{[1]}]] \subseteq R^{[0]}[z^{[01]}] \\
\text{iff} & R \circ_E R \subseteq R
\end{array}
$$

$$x(R \circ_E S)a \quad \text{iff} \quad \exists b(xRb \ \&\ E^{(1)}[b] \subseteq S^{(0)}[a]).$$

Epistemic interpretation of modal axioms

Axiom	Kripke frames	Graph-based frames
$\Box p \to p$ Factivity: if agent knows p then p true	$\Delta \subseteq R$ agent can tell apart only non-identical states	$E \subseteq R$ agent can tell apart only non-inher. indist. states
$\Box p \to \Box\Box p$ Positive introspection: if agent knows p then agent knows of knowing p	$R \circ R \subseteq R$ if agent tells apart x, y then agent can distinguish y from any z agent cannot tell apart from x	$R \circ_E R \subseteq R$ positive introspection + inherent indistinguishab.

A last example

p: 'the defendant has not willingly caused harm to her friend'

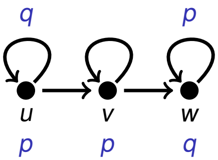
q: 'the defendant acted in self-defence'

The defendant is not guilty if and only if $p \vee q$.

u: "I saw her grabbing a tennis racket and hitting her friend. She looked terrified."

v: "I saw her grabbing a tennis racket and hitting her friend. She looked frightened, but not necessarily by her friend."

w: "I heard her scream that there was a poisonous spider on her friend's shoulder, so she killed the spider."



There is no witness that provides enough evidence to refute both p and q, hence, all testimonies lead to the acceptance of a not guilty verdict.