Deduction via 2-category theory

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$$(\mathrm{Cut})\ \frac{x;\Gamma \vdash \phi \qquad x;\Gamma,\phi \vdash \psi}{x;\Gamma \vdash \psi}$$

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We have two different deductive systems doing similar things.

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Can category theory help?

(Some) categorical models

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Categories with families[1],
natural models[2], …

$$\begin{tikzcd} \dot{\mathbb{U}} \arrow[rr] \arrow[dr, "fib"'] & & \mathbb{U} \arrow[dl, "fib"] \\ & \mathrm{ctx} & \end{tikzcd}$$

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Doctrines and hyperdoctrines[3], …

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[2] Awodey, "Natural models of homotopy type theory", 2018.
[3] Lawvere, "Adjointness in Foundations", 1969.

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& A \arrow[r, "\phi"'] \arrow[dd, mapsto] & B \arrow[dd, mapsto] \\
Z \arrow[dr, "h"'] \arrow[drr, bend left, "g"] & & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

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We build a theory were:

judgements = functors (fibrations)
rules = (lax) commutative triangles

An account of context, judgement, deduction

context

judgement

deduction

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A *pre-judgemental theory* is specified through the following data:

context

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context (ctx) a category (with terminal object $\diamond$);

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judgement ($\mathcal{J}$) *judgement classifiers*, a class of functors $f : \mathbb{F} \to \mathrm{ctx}$ over the category of contexts, possibly (op)fibrations;

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deduction ($\mathcal{R}$) *rules*, a class of functors $\lambda : \mathbb{F} \to \mathbb{G}$;
($\mathcal{P}$) *policies*, a class of 2-dimensional cells filling (some) triangles induced by rules (functors in $\mathcal{R}$) and judgements (functors in $\mathcal{J}$).

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Why fibrations? - reprise

$$p : \mathbb{E} \longrightarrow \mathbb{B}$$

They are functors
$\rightsquigarrow$ great for encoding dependencies:

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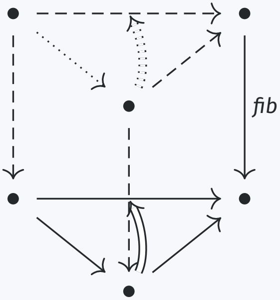
Lemma (♯-lifting)



Categories as syntax

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Whenever $F, F' \in f^{-1}(\Gamma)$ and $F = F'$ we read $\Gamma \vdash F =_{\mathbb{F}} F'$.

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and, possibly, Γ and $g\lambda F$ and related by a map

$$\lambda^{\sharp}_F : g\lambda F \to \Gamma$$

Example: toy MLTT

toy MLTT: $\begin{cases} \text{ctx : (the syntactic category of) contexts and substitutions} \\ \mathcal{J} = \{\dot{u}, u\} \\ \mathcal{R} = \{\Sigma\} \\ \mathcal{P} = \{id : u \circ \Sigma \Rightarrow \dot{u}\} \end{cases}$

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Università di Genova ItaCa

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Then

- 2 dimensions are necessary;
- 2 dimensions are sufficient!*

* Provided that the ambient 2-category has some structure. Here: **Cat**.

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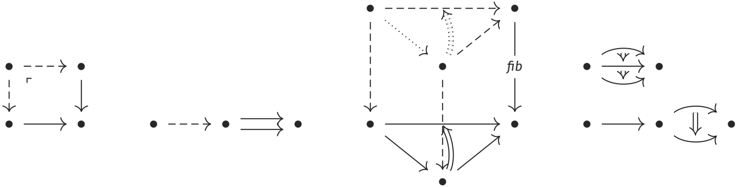
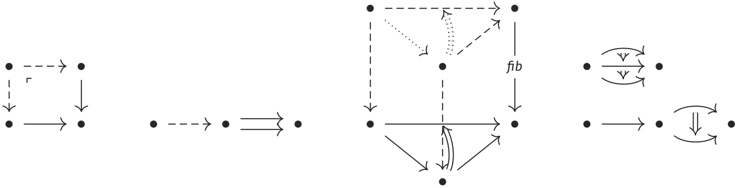


Judgemental theories

A *judgemental theory* (ctx, $\mathcal{J}, \mathcal{R}, \mathcal{P}$) is a pre-judgemental theory such that

1. $\mathcal{R}$ and $\mathcal{P}$ are closed under composition;
2. the judgements are precisely those rules whose codomain is ctx;
3. $\mathcal{R}$ and $\mathcal{P}$ are closed under *finite limits*, ♯*-liftings*, *whiskering* and *pasting*.



We now have a calculus!

Judgemental theories: the motto

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Every rule is a diagram.

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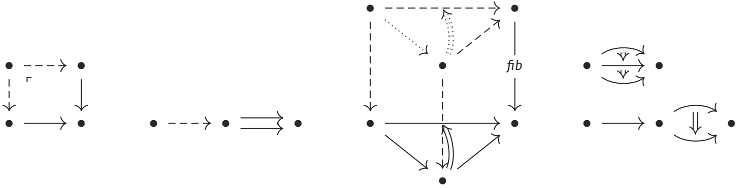

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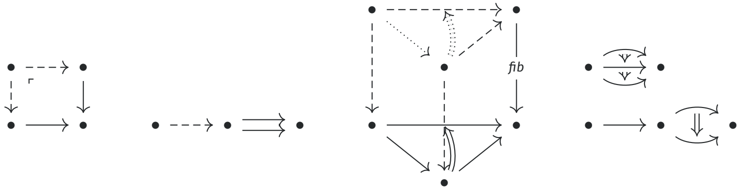

Every diagram is a rule.

$\rightsquigarrow$ any triangle we find in our jt is a rule we *prove*

Nested judgements

Pullbacks compute *nested judgements* such as

$$\Gamma \vdash a : A \qquad \Gamma.A \vdash B$$

$$x; \Gamma \vdash \phi \qquad x; \Gamma, \phi \vdash \psi$$

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because

$$\begin{tikzcd}
\bullet \arrow[dr, dashed, "F\lambda.H"] \arrow[drr, dashed, bend left, "H"] \arrow[dddr, dashed, bend right, "F"'] & & \\
& \mathbb{F}\lambda.\mathbb{H} \arrow[r, dotted, "f.g\lambda"] \arrow[dd, dotted, "g\lambda.f"] \arrow[ddr, phantom, very near start, "\ulcorner"] & \mathbb{H} \arrow[d, "\lambda"] \\
& & \mathbb{G} \arrow[d, "g"] \\
& \mathbb{F} \arrow[r, "f"] & \mathrm{ctx}
\end{tikzcd}$$

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really is

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The jt of natural deduction

$$p : \mathrm{ctx}^{op} \longrightarrow \mathrm{Pos}$$

s.t. it has fibered products preserved by reindexing

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Examples

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$$\text{jND:} \quad \begin{cases} \mathsf{ctx} : \text{contexts and substitutions } \textit{e.g. } \mathbf{Fin} \\ \mathcal{J} = \{p\} \text{ s.t. faithful, with fibered products} \\ \mathcal{R} = \dots \\ \mathcal{P} = \dots \end{cases}$$

The jt of natural deduction

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[r, "- \wedge -"] \arrow[d, "p^2"'] & \mathbb{P} \arrow[d, "p"] \\
\mathrm{ctx}^2 \arrow[r, "- \times -"] \arrow[r, phantom, shift right=2, "\perp"] & \mathrm{ctx} \arrow[l, bend left, "diag"]
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then close under finite limits, ♯-lifting, ...

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Weakening

$$
\begin{tikzcd}
\mathbb{P}^{\times} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P} \arrow[d, "p"] \\
\mathrm{ctx}^2 \arrow[r, "- \times -"] & \mathrm{ctx}
\end{tikzcd}
$$

Weakening

$$\begin{tikzcd}
\mathbb{P}^2 \arrow[drr, bend left, "- \wedge -"] \arrow[ddr, bend right, "p^2"'] & & \\
& \mathbb{P}^\times \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P} \arrow[d, "p"] \\
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Weakening

$$
\begin{tikzcd}
\mathbb{P}^2 \arrow[dr, dashed, "w"'] \arrow[drr, bend left, "- \wedge -"] \arrow[ddr, bend right, "p^2"'] & & \\
& \mathbb{P}^\times \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P} \arrow[d, "p"] \\
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& \mathrm{ctx}^2 \arrow[r, "-\times-"] & \mathrm{ctx} & x & y
\end{tikzcd}$$

Weakening

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\begin{tikzcd}
\mathbb{P}^2 \arrow[dr, dashed, "w"] \arrow[drr, bend left, "- \wedge -"] \arrow[ddr, bend right, "p^2"'] & & \\
& \mathbb{P}^\times \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{P} \arrow[d, "p"] \\
& \mathrm{ctx}^2 \arrow[r, "- \times -"] & \mathrm{ctx}
\end{tikzcd}
\qquad
\begin{tikzcd}
\phi & & \psi \\
& x \times y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
x & & y
\end{tikzcd}
$$

Weakening

$$
\begin{tikzcd}
\mathbb{P}^2 \arrow[dr, dashed, "w"] \arrow[drr, bend left, "- \wedge -"] \arrow[ddr, bend right, "p^2"'] & & \\
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& \mathrm{ctx}^2 \arrow[r, "- \times -"] & \mathrm{ctx}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& w(\phi, \psi) = \phi[p_1] \wedge \psi[p_2] \arrow[dl] \arrow[dr] & \\
\phi & & \psi
\end{tikzcd}
$$

$$
\begin{tikzcd}
& x \times y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
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$$(w)\ \frac{x \times y \vdash (\phi, \psi)\ \mathbb{P}^2}{x \times y \vdash \phi[p_1] \wedge \psi[p_2]\ \mathbb{P}^\times}$$

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Stratified contexts

$$x; \Gamma \vdash \psi$$

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$$\begin{tikzcd}
\mathbb{P}^{\rightarrow} I.p^{\rightarrow}\mathsf{ctx} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{ctx} \arrow[d, "Id"] \\
\mathbb{P}^{\rightarrow} \arrow[r, "p^{\rightarrow}"] \arrow[d, shift right, "\mathsf{cod}"'] \arrow[d, shift left, "\mathsf{dom}"] & \mathsf{ctx}^{\rightarrow} \arrow[d, shift right, "\mathsf{cod}"'] \arrow[d, shift left, "\mathsf{dom}"] \\
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\end{tikzcd}$$

Stratified contexts

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$$\begin{tikzcd}
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\mathbb{P}^{\to} \arrow[r, "p^{\to}"] \arrow[d, shift right] \arrow[d, shift left] & \mathrm{ctx}^{\to} \arrow[d, shift right, "\mathrm{cod}"'] \arrow[d, shift left, "\mathrm{dom}"] \\
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\end{tikzcd}$$

Stratified contexts

$$x; \Gamma \vdash \psi$$

Remark

e is a fibration.

$$\begin{tikzcd}
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$$\begin{tikzcd}
{\color{red}\mathbb{E}} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[dd, bend right, red, "c"'] \arrow[dd, bend left, red, "d"] \arrow[ddr, red, "e"] & \mathsf{ctx} \arrow[d, "Id"] \\
\mathbb{P}^{\to} \arrow[r, "p^{\to}"] \arrow[d, shift right] \arrow[d, shift left] & \mathsf{ctx}^{\to} \arrow[d, shift right, "\mathsf{cod}"'] \arrow[d, shift left, "\mathsf{dom}"] \\
\mathbb{P} \arrow[r, "p"] & \mathsf{ctx}
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$x; \Gamma \vdash \psi$ iff

Stratified contexts

$$x; \Gamma \vdash \psi$$

Remark

e is a fibration.

$$x; \Gamma \vdash \psi \text{ iff } \begin{cases} x \vdash e\ \mathbb{E} \\ x \vdash \mathrm{dom}(e) =_{\mathbb{P}} \bigwedge \Gamma \\ x \vdash \mathrm{cod}(e) =_{\mathbb{P}} \psi \end{cases}$$

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From structure to rules

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Proposition

The following rule is in jND.

$$(T)\ \frac{x;\psi\vdash\phi \qquad x;\phi\vdash\chi}{x;\psi\vdash\chi}$$

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\mathbb{E}d.c\mathbb{E} \arrow[rrrr, dashed] \arrow[drr, dotted] \arrow[dd, dashed] & & & & \mathbb{E} \arrow[dd, "d"] \\
& & \mathbb{E}d.d\mathbb{E} \arrow[urr, dashed] \arrow[dd, dashed] \arrow[u, phantom, "\Uparrow"] & & \\
\mathbb{E} \arrow[rrrr, "c"] \arrow[drr, "id"'] & & & & \mathbb{P} \arrow[drr, "p"] & & \\
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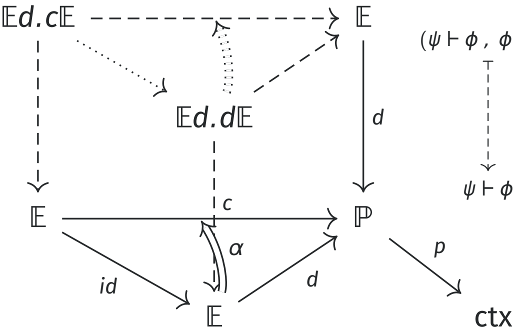

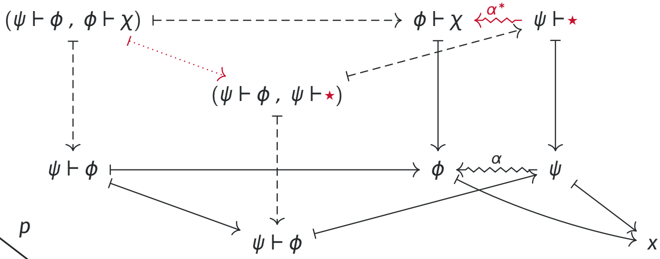

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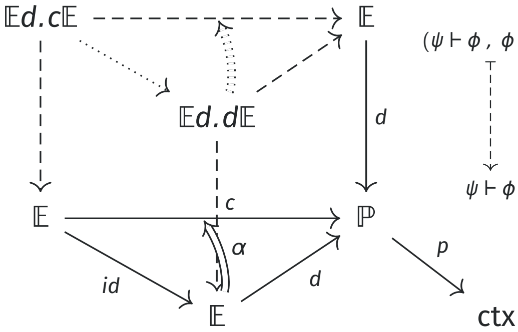

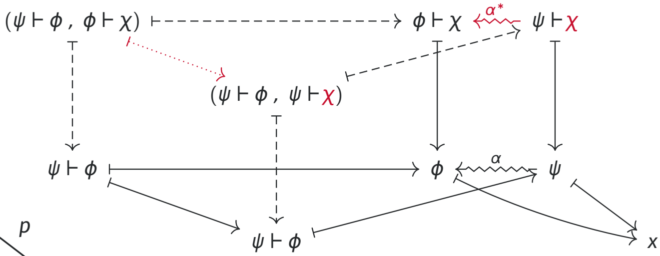

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& \mathrm{ctx} &
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$$(\psi\vdash\phi\,,\ \phi\vdash\chi)$$

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$(\psi\vdash\phi\ ,\ \phi\vdash\chi)$ $\qquad$ $(\psi\vdash\phi\ ,\ \psi\vdash\chi)$

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x

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\mathbb{E}d.c\mathbb{E} \arrow[rr, "!"] \arrow[dr] & & \mathbb{E}d.d\mathbb{E} \arrow[dl] \arrow[ll, phantom, "\Leftarrow"] \\
& \mathbb{E} \arrow[d, "d"] & \\
& \mathbb{P} \arrow[d, "p"] & \\
& \mathrm{ctx} &
\end{tikzcd}$$

$(\psi \vdash \phi\ ,\ \phi \vdash \chi)$ $\qquad$ $(\psi \vdash \phi\ ,\ \psi \vdash \chi)$

$\phi \vdash \chi \leftarrow \psi \vdash \chi$

$\phi \geq \psi$

x

Cut elimination theorem

Theorem

The following rule is in jND.

$$(Cut)\ \frac{x; \Gamma \vdash \phi \qquad x; \Gamma, \phi \vdash \chi}{x; \Gamma \vdash \chi}$$

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$$S : \mathbb{E} \longrightarrow \mathbb{E}, \quad (\psi \leq \phi) \longmapsto (\psi \wedge \phi \leq \psi \leq \phi)$$

part of a monad related to the simple fibration

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$$
\begin{prooftree}
\AxiomC{$x;\Gamma\vdash\phi$}
\AxiomC{$x;\Gamma,\phi\vdash\psi$}
\BinaryInfC{$x;\Gamma\vdash\phi$}
\UnaryInfC{$x;\Gamma\vdash\phi_\Gamma\wedge\phi$}
\AxiomC{$x;\Gamma\vdash\phi$}
\AxiomC{$x;\Gamma,\phi\vdash\psi$}
\BinaryInfC{$x;\Gamma,\phi\vdash\psi$}
\doubleLine
\UnaryInfC{$x;\phi_\Gamma\wedge\phi\vdash\psi$}
\LeftLabel{$(\top)$}
\BinaryInfC{$x;\Gamma\vdash\psi$}
\end{prooftree}
$$

$$
\begin{tikzcd}
\mathbb{E}d.dS\mathbb{E} \arrow[d, "d.dS"'] \arrow[dr, dashed] \arrow[drr, bend left, "dS.d"] & & \\
\mathbb{E} \arrow[d, "(id,d)"'] & \mathbb{E}d.c\mathbb{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d, dashed, "d"] \\
\mathbb{E}\times\mathbb{P} \arrow[r] & \mathbb{E} \arrow[r, "c"] & \mathbb{P}
\end{tikzcd}
$$

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& \color{red}{\mathbb{E}d.c\mathbb{E}} \arrow[rr, "(T)"] \arrow[dr] & & \mathbb{E}d.d\mathbb{E} \arrow[dl] \arrow[ll, Rightarrow, shorten=1em] \\
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$$\begin{tikzcd}
\dot{\mathbb{U}}.\Delta\Sigma\mathbb{U} \arrow[rr] \arrow[dr, bend right] & & \dot{\mathbb{U}} \times \mathbb{U} \arrow[dl, bend left] \arrow[dll, phantom, "\Leftarrow"] \\
& \mathbb{U} &
\end{tikzcd}$$

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$$\text{(DTy)}\ \frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B}{\Gamma \vdash B[a]} \qquad\qquad \text{(Cut)}\ \frac{x;\Gamma \vdash \phi \qquad x;\Gamma,\phi \vdash \psi}{x;\Gamma \vdash \psi}$$

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… plus both $\Delta\Sigma$ and S are monads!

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Thank you for listening!

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