# Lambek-Grishin Calculus:
# Focusing, Display and Full Polarization

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TACL 2022, Coimbra
23 June

# Identity of proofs

Identifying or telling apart proofs has far-reaching consequences.

- Philosophy and mathematics: when do two proofs correspond to the same argument?
- Computer science: when do two algorithms correspond to the same program?
- Linguistics: how to capture different readings of the same sentence?
- ...

**Sequent calculi** exhibit syntactically different proofs of the very same end-sequent often due to **trivial permutations** of inference rules.

**Natural deduction calculi** or **proof nets** are less sensitive to inference rule permutations and are taken as **benchmarks** for defining identity of proofs.

**Focused sequent calculi** [And92, And01, Mil04] make use of syntactic restrictions on the applicability of inference rules achieving three main goals:

1. the proof search space is reduced retaining **completeness**;
2. every cut-free proof comes in a **special normal form**;
3. criterion for defining **identity of sequent calculi proofs**.

What is the mathematical underpinning of focalization?

Looking for:

- (uniform and modular structural) **proof theory** and
- (algebraic and categorical) **semantics**.

# Our contributions

Lambek-Grishin logic (expanded with analytic structural rules closed under **mutations**, i.e. an equivalence relation between structural connectives: see Appendix A.):

- (heterogenous multi-type) focused display calculus **fD.LG**
- fully polarized algebraic semantics $\mathbb{FP}.\mathbb{LG}$

where:

- **fD.LG** has canonical cut-elimination, strong focalization, and is complete w.r.t. $\mathbb{FP}.\mathbb{LG}$
- **fD.LG** is complete w.r.t. LG-algebras $\rightsquigarrow$ semantic proof of completeness of focusing (given that the standard display calculus for LG is complete w.r.t. LG-algebras)
- effective translation between **fD.LG**-proofs and **fLG**-proofs [MM12] $\rightsquigarrow$ operational semantics (given that **fLG**-derivations are in a Curry-Howard correspondence with directional $\overline{\lambda}\mu\widetilde{\mu}$-terms)

General theory:

- heterogenous multi-type display calculi
- fully polarized algebras

We expect that the approach extends to every displayable logic (see Conclusions).

# Basic Lambek-Grishin logic 1/2

**Basic Lambek-Grishin algebra** [Moo09]:

- Poset $\mathbb{G} = (G, \leq)$
- 6 operations $\otimes, \oplus, \backslash, \obslash, /, \oslash$ s.t.

$$
\begin{array}{ccccc}
B \leq A \backslash C & \text{iff} & A \otimes B \leq C & \text{iff} & A \leq C / B \\
C \oslash B \leq A & \text{iff} & C \leq A \oplus B & \text{iff} & A \obslash C \leq B
\end{array} \tag{1}
$$

$$
\frac{\text{John}}{np} \otimes \frac{\text{sleeps}}{np \backslash s} \ \overset{\text{is}}{\leq} \ \frac{\text{a sentence}}{s}
$$

# Proper Multi-type Display Calculi

- Natural generalization of Gentzen's sequent calculi;
- **Display property**:

$$\frac{\dfrac{Y \vdash X > Z}{X \,;\, Y \vdash Z}}{X \vdash Z < Y}$$

  display rules semantically justified by **adjunction/residuation**

- **Multi-type:** Separate **syntactic types** for different types of semantic objects
- **Proper:** Rules closed under **uniform substitution** (Wansing '98) within each type
- **Canonical proof of cut elimination (via metatheorem)**

# Multi-type proper display calculi

[Greco et al. 14...]

## Definition

A **proper DC** verifies each of the following conditions:

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined *congruence* relation (same shape, position, non-proliferation)
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters within each type (Properness!);
5. **reduction strategy** exists when cut formulas are principal.
6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

## Theorem (Canonical!)

Cut elimination and subformula property hold for any **proper m.DC**.

# Basic Lambek-Grishin logic 2/2

**D.LG** consists of the following rules (we consider only the Lambek fragment for brevity).

**Axioms and cuts:**

$$\frac{}{p \vdash p}\ \text{Id} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \text{Cut}$$

**Logical rules** (i.e. **translation** vs **tonicity rules**, cfr. asynchronous vs synchronous [And01]):

$$\otimes_L\ \frac{A \mathbin{\hat{\otimes}} B \vdash X}{A \otimes B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \mathbin{\hat{\otimes}} Y \vdash A \otimes B}\ \otimes_R$$

$$\backslash_L\ \frac{X \vdash A \qquad B \vdash Y}{A \backslash B \vdash X \mathbin{\check{\backslash}} Y} \qquad \frac{X \vdash A \mathbin{\check{\backslash}} B}{X \vdash A \backslash B}\ \backslash_R \qquad /_L\ \frac{B \vdash Y \qquad X \vdash A}{B / A \vdash Y \mathbin{\check{/}} X} \qquad \frac{X \vdash B \mathbin{\check{/}} A}{X \vdash B / A}\ /_R$$

**Display postulates**:

$$\begin{array}{rc} & Y \vdash X \mathbin{\check{\backslash}} Z \\ \hat{\otimes} \dashv \check{\backslash} & \overline{\overline{\phantom{X \mathbin{\hat{\otimes}} Y \vdash Z}}} \\ & X \mathbin{\hat{\otimes}} Y \vdash Z \\ \hat{\otimes} \dashv \check{/} & \overline{\overline{\phantom{X \mathbin{\hat{\otimes}} Y \vdash Z}}} \\ & X \vdash Z \mathbin{\check{/}} Y \end{array}$$

We may expand the calculus with so-called **Structural rules**, e.g.:

$$\frac{(X \mathbin{\hat{\otimes}} Y) \mathbin{\hat{\otimes}} Z \vdash W}{\overline{X \mathbin{\hat{\otimes}} (Y \mathbin{\hat{\otimes}} Z) \vdash W}}\ a, a^{-1}$$

# Everybody needs somebody

$$\frac{\text{Everybody}}{s \,/\, (np \,\backslash\, s)} \otimes \frac{\text{needs}}{(((np \,\backslash\, s) \,/\, ((s \,/\, np) \,\backslash\, s))} \otimes \frac{\text{somebody}}{(s \,/\, np) \,\backslash\, s} \quad \underset{\vdash}{\text{is}} \quad \frac{\text{a sentence}}{s}$$

There are 7 different sequent derivations, but only 3 different natural deduction (or proof net) derivations (in normal form).

Moving to a focused sequent system (**fLG** or **fD.LG**) we have again 3 derivations in normal form (with the bias assignment *np*:: positive, *s*:: negative).

Two derivations use associativity and correspond to the following readings:

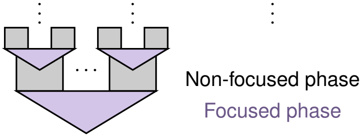
- ∀-∃ reading: Everybody > somebody > needs
- ∃-∀ reading: Somebody > everybody > needs

# Focalization (1/2)

The key idea relies on the following distinction.

- A **focused phase** is a proof-section where a formula is decomposed "as much as possible" only by means of non-invertible logical rules. This formula and all its immediate subformulas in this proof-section are said 'in focus'.
- A **neutral phase** is a non-focused phase, i.e. a proof section built by translation rules (applied greedily) or structural rules.

A **strongly focalized proof** exhibits a strict alternation between focused and neutral phases:




## Definition

A sequent proof $\pi$ is **strongly focalized** if cut-free and, for every formula $A$ occurring in $\pi$, every **PIA subtree** of $A$ is constructed by a proof-section of $\pi$ containing only **tonicity rules**.

# Focalization via polarization

Two focalized phases:

- **positive phase**: only non-invertible logical rules for positive connectives are applied;
- **negative phase**: only non-invertible logical rules for negative connectives are applied.

How to categorize a connective as "positive" or "negative"?
The usual answer has to do with the distinction: "right" versus "left" logical rules.

The mathematical underpinning is the following:

- **Positive formulas**: the main connective is a left-adjoint/residual (LG: $\otimes, \oslash, \obslash$);
- **Negative formulas**: the main connective is a right-adjoint/residual (LG: $\oplus, \backslash, /$).

The key idea of polarization "naturally" calls for a type distinction.
So, multi-type calculi seem a good candidate. . . but we need a further generalization.

# A step back: focalization via "implicit" polarization

State of the art: **fLG** by Moortgat and Moot 2011 [MM12]

- Every proof is strongly focalized
- Focus implemented by a meta-linguistical marker $\boxed{A}$
- **Restrictions on the applicability of rules**

If $A$ is positive:

| Axiom | Focusing | Defocusing |
|---|---|---|
| $\dfrac{}{A \vdash \boxed{A}}$ | $\leftharpoondown \dfrac{A \vdash \Delta}{\boxed{A} \vdash \Delta}$ | $\dfrac{X \vdash \boxed{A}}{X \vdash A} \rightharpoonup$ |

If $A$ is negative:

| Co-axiom | Focusing | Defocusing |
|---|---|---|
| $\dfrac{}{\boxed{A} \vdash A}$ | $\dfrac{X \vdash A}{X \vdash \boxed{A}} \rightharpoondown$ | $\leftharpoonup \dfrac{\boxed{A} \vdash X}{A \vdash X}$ |

Tonicity rules have auxiliary and principal formulas in focus.

# ∀-∃ reading: **fLG**

Bias assignment: *np* :: positive, *s* :: negative.

$$\dfrac{\boxed{s_5} \vdash s_7 \qquad np_9 \vdash \boxed{np_6}}{\boxed{s \,/\, np} \vdash s \,\check{/}\, np}$$

$$\dfrac{\boxed{s \,/\, np} \vdash s \,\check{/}\, np}{np \vdash \boxed{(s \,/\, np) \,\backslash\, s}} \;\rightleftharpoons$$

$$\dfrac{np_1 \vdash \boxed{np_3} \qquad \boxed{s_4} \vdash s_8}{\boxed{np \,\backslash\, s} \vdash np \,\check{\backslash}\, s}$$

$$\dfrac{\boxed{np \,\backslash\, s} \vdash np \,\check{\backslash}\, s \qquad np \vdash \boxed{(s \,/\, np) \,\backslash\, s}}{\boxed{(np \,\backslash\, s) \,/\, ((s \,/\, np) \,\backslash\, s)} \vdash (np \,\check{\backslash}\, s) \,\check{/}\, np}$$

$$\dfrac{\boxed{(np \,\backslash\, s) \,/\, ((s \,/\, np) \,\backslash\, s)} \vdash (np \,\check{\backslash}\, s) \,\check{/}\, np}{np \,\hat{\otimes}\, ((np \,\backslash\, s) \,/\, ((s \,/\, np) \,\backslash\, s)) \vdash \boxed{s \,/\, np}} \;\rightleftharpoons, a$$

$$\dfrac{np \,\hat{\otimes}\, ((np \,\backslash\, s) \,/\, ((s \,/\, np) \,\backslash\, s)) \vdash \boxed{s \,/\, np} \qquad \boxed{s_{10}} \vdash s_2}{\boxed{(s_8 \,\backslash\, np_9) \,\backslash\, s_{10}} \vdash (np \,\hat{\otimes}\, ((np \,\backslash\, s) \,/\, (s \,/\, np) \,\backslash\, s))) \,\check{\backslash}\, s}$$

$$\dfrac{\boxed{(s_8 \,\backslash\, np_9) \,\backslash\, s_{10}} \vdash (np \,\hat{\otimes}\, ((np \,\backslash\, s) \,/\, (s \,/\, np) \,\backslash\, s))) \,\check{\backslash}\, s}{((np \,\backslash\, s) \,/\, ((s \,/\, np) \,\backslash\, s)) \,\hat{\otimes}\, ((s \,/\, np) \,\backslash\, s) \vdash \boxed{np \,\backslash\, s}} \;\rightleftharpoons, a^{-1}$$

$$\dfrac{\boxed{s_0} \vdash s_{11} \qquad ((np \,\backslash\, s) \,/\, ((s \,/\, np) \,\backslash\, s)) \,\hat{\otimes}\, ((s \,/\, np) \,\backslash\, s) \vdash \boxed{np \,\backslash\, s}}{\boxed{s_0 \,/\, (np_1 \,\backslash\, s_2)} \vdash s \,\check{/}\, (((np \,\backslash\, s) \,/\, ((s \,/\, np) \,\backslash\, s)) \,\hat{\otimes}\, ((s \,/\, np) \,\backslash\, s))}$$

$$\dfrac{\boxed{s_0 \,/\, (np_1 \,\backslash\, s_2)} \vdash s \,\check{/}\, (((np \,\backslash\, s) \,/\, ((s \,/\, np) \,\backslash\, s)) \,\hat{\otimes}\, ((s \,/\, np) \,\backslash\, s))}{\underbrace{s_0 \,/\, (np_1 \,\backslash\, s_2)}_{\textit{Everybody}} \,\hat{\otimes}\, (\underbrace{((np_3 \,\backslash\, s_4) \,/\, ((s_5 \,/\, np_6) \,\backslash\, s_7))}_{\textit{needs}} \,\hat{\otimes}\, \underbrace{((s_8 \,/\, np_9) \,\backslash\, s_{10})}_{\textit{somebody}}) \vdash \boxed{s_{11}}} \;\rightleftharpoons$$

Bias assignment: *np* :: positive, *s* :: negative.

$$\dfrac{}{\boxed{s_5} \vdash s_7} \qquad \dfrac{}{np_9 \vdash \boxed{np_6}}$$

$$\dfrac{\boxed{s_5} \vdash s_7 \qquad np_9 \vdash \boxed{np_6}}{\boxed{s \,/\, np} \vdash s \,\check{/}\, np}$$

$$\boxed{s \,/\, np} \vdash s \,\check{/}\, np \;\;\dashrightarrow\;\; np \vdash \boxed{(s \,/\, np) \backslash s} \quad \leftrightharpoons$$

$$\dfrac{}{np_1 \vdash \boxed{np_3}} \qquad \dfrac{}{\boxed{s_4} \vdash s_2}$$

$$\dfrac{np_1 \vdash \boxed{np_3} \qquad \boxed{s_4} \vdash s_2}{\boxed{np \backslash s} \vdash np \,\check{\backslash}\, s}$$

$$\dfrac{\boxed{np \backslash s} \vdash np \,\check{\backslash}\, s \qquad np \vdash \boxed{(s \,/\, np) \backslash s}}{\boxed{(np \backslash s) \,/\, ((s \,/\, np) \backslash s)} \vdash (np \,\check{\backslash}\, s) \,\check{/}\, np}$$

$$\boxed{(np \backslash s) \,/\, ((s \,/\, np) \backslash s)} \vdash (np \,\check{\backslash}\, s) \,\check{/}\, np \;\;\dashrightarrow\;\; ((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \,\hat{\otimes}\, np \vdash \boxed{np \backslash s} \quad \leftrightharpoons$$

$$\dfrac{}{\boxed{s_0} \vdash s_8}$$

$$\dfrac{\boxed{s_0} \vdash s_8 \qquad ((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \,\hat{\otimes}\, np \vdash \boxed{np \backslash s}}{\boxed{s_0 \,/\, (np_1 \backslash s_2)} \vdash s \,\check{/}\, (((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \,\hat{\otimes}\, np)}$$

$$\boxed{s_0 \,/\, (np_1 \backslash s_2)} \vdash s \,\check{/}\, (((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \,\hat{\otimes}\, np) \;\;\dashrightarrow\;\; (s_0 \,/\, (np_1 \backslash s_2)) \,\hat{\otimes}\, ((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \vdash \boxed{s \,/\, np} \quad \leftrightharpoons, a$$

$$\dfrac{}{\boxed{s_{10}} \vdash s_{11}}$$

$$\dfrac{(s_0 \,/\, (np_1 \backslash s_2)) \,\hat{\otimes}\, ((np \backslash s) \,/\, ((s \,/\, np) \backslash s)) \vdash \boxed{s \,/\, np} \qquad \boxed{s_{10}} \vdash s_{11}}{\boxed{(s_8 \,/\, np_9) \backslash s_{10}} \vdash ((s \,/\, (np \backslash s)) \,\hat{\otimes}\, ((np \backslash s) \,/\, ((s \,/\, np) \backslash s))) \,\check{\backslash}\, s}$$

$$\boxed{(s_8 \,/\, np_9) \backslash s_{10}} \vdash ((s \,/\, (np \backslash s)) \,\hat{\otimes}\, ((np \backslash s) \,/\, ((s \,/\, np) \backslash s))) \,\check{\backslash}\, s \;\;\dashrightarrow\;\; \underbrace{s_0 \,/\, (np_1 \backslash s_2)}_{\textit{Everybody}} \,\hat{\otimes}\, (\underbrace{((np_3 \backslash s_4) \,/\, ((s_5 \,/\, np_6) \backslash s_7))}_{\textit{needs}} \,\hat{\otimes}\, \underbrace{((s_8 \,/\, np_9) \backslash s_{10})}_{\textit{somebody}}) \vdash \boxed{s_{11}} \quad \leftrightharpoons, a^{-1}$$

# Focalization via "explicit" polarization

In the proof-theoretical literature, so-called **shifts** "operator" have been considered.

The key idea is the following:

- if A is negative, then $\downarrow A$ is positive;
- if A is positive, then $\uparrow A$ is negative.

But their status as operators is obscure.

On the other hand, in algebraic/categorical polarized semantics we have:

- $\uparrow \dashv \downarrow$;
- $\uparrow\downarrow\uparrow \varphi \; = \uparrow \varphi$; $\downarrow\uparrow \varphi \; = \; \varphi$;
- $\downarrow\uparrow\downarrow \varphi \; = \downarrow \varphi$; $\uparrow\downarrow \varphi \; = \; \varphi$.

Problem: the focusing policy could be destroyed.
The usual solution is to consider only sequents where $\uparrow$ (resp. $\downarrow$) does not immediately occurr under the scope of $\downarrow$ (resp. $\uparrow$).

Our solution: we distinguish between positive (resp. negative) **pure** formulas and positive (resp. negative) **shifted** formulas, i.e. formulas under the scope of a shift operator.

# Weakening relations

W.R. are the **order-theoretic equivalents** of **profunctors** (aka distributors or bimodules) [Ben73].

W.R. are **generalizations** of **partial orders**: take $\mathcal{A} = \mathcal{B}$ and $\leq_{\mathcal{A}} = \leq_{\mathcal{B}}$.

## Definition

A **weakening relation** is a relation $\preccurlyeq \subseteq \mathcal{A} \times \mathcal{B}$ on two partially ordered set $(\mathcal{A}, \leq_{\mathcal{A}})$ and $(\mathcal{B}, \leq_{\mathcal{B}})$ that is **compatible with the orders** $\leq_{\mathcal{A}}$ and $\leq_{\mathcal{B}}$ in the following sense

$$\frac{A' \leq_{\mathcal{A}} A \qquad A \preccurlyeq B \qquad B \leq_{\mathcal{B}} B'}{A' \preccurlyeq B'}$$

## Definition

Given two w.r. $\preccurlyeq_{\mathcal{A}} \subseteq \mathcal{A} \times \mathcal{A}'$ and $\preccurlyeq_{\mathcal{B}} \subseteq \mathcal{B} \times \mathcal{B}'$, we say that the order-preserving functions $L : \mathcal{A} \to \mathcal{B}$ and $R : \mathcal{B}' \to \mathcal{A}'$ form a **heterogeneous adjoint pair** $L \dashv^{\preccurlyeq_{\mathcal{B}}}_{\preccurlyeq_{\mathcal{A}}} R$ if for every $A \in \mathcal{A}$ and $B' \in \mathcal{B}'$,

$$L(A) \preccurlyeq_{\mathcal{B}} B' \text{ iff } A \preccurlyeq_{\mathcal{A}} R(B')$$

$$\begin{tikzcd} \mathcal{A}' & & \mathcal{B}' \arrow[ll, bend right, "R"'] \\ \mathcal{A} \arrow[u, "\preccurlyeq_{\mathcal{A}}"] \arrow[rr, bend right, "L"'] \arrow[urr, phantom, "\top"] & & \mathcal{B} \arrow[u, "\preccurlyeq_{\mathcal{B}}"'] \end{tikzcd}$$

If $\mathcal{A}' = \mathcal{A}$, $\preccurlyeq_{\mathcal{A}} = \leq_{\mathcal{A}}$, $\mathcal{B}' = \mathcal{B}$ and $\preccurlyeq_{\mathcal{B}} = \leq_{\mathcal{B}}$, we recover the usual definition of adjunction.

Heterogeneous adjunctions also appear in the theory of Chu spaces.

# Full polarization (1/2)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\dot{\mathbb{P}} \arrow[loop left, "{\cdot\preceq\cdot}^{+}"] \arrow[r, "{\cdot\preceq}", "|" marking] & \mathbb{N} \arrow[loop right, "\preceq^{-}"] \\
\mathbb{P} \arrow[loop left, "\preceq^{+}"] \arrow[u, "{\preceq\cdot}^{+}", "-" marking] \arrow[ur, "\preceq", "|" marking] \arrow[r, "{\preceq\cdot}"', "|" marking] & \dot{\mathbb{N}} \arrow[loop right, "{\cdot\preceq\cdot}^{-}"] \arrow[u, "{\cdot\preceq}^{-}"', "-" marking]
\end{tikzcd}
$$

---

$$
\begin{tikzcd}[column sep=huge]
\mathring{\mathbb{P}} \arrow[loop left, "\mathring{\preceq}^{+}"] \arrow[r, "\mathring{\preceq}", "|" marking] & \mathring{\mathbb{N}} \arrow[loop right, "\mathring{\preceq}^{-}"]
\end{tikzcd}
$$

(Heterogeneous) **operations** and their residuals (we consider the Lambek fragment for brevity):

$$\otimes : \mathring{\mathbb{P}} \times \mathring{\mathbb{P}} \to \mathbb{P} \qquad \backslash : \mathring{\mathbb{P}}^{\partial} \times \mathring{\mathbb{N}} \to \mathbb{N} \qquad / : \mathring{\mathbb{N}} \times \mathring{\mathbb{P}}^{\partial} \to \mathbb{N}$$
$$\mathring{Q} \mathrel{\mathring{\lesssim}} \mathring{P} \backslash \mathring{N} \quad \text{iff} \quad \mathring{P} \otimes \mathring{Q} \mathrel{\mathring{\lesssim}} \mathring{N} \quad \text{iff} \quad \mathring{P} \mathrel{\mathring{\lesssim}} \mathring{N} / \mathring{Q} \tag{2}$$

**Shifts**:

$$\begin{tikzcd} \mathbb{P} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \dot{\mathbb{N}} \arrow[l, bend left, "\downarrow"] \end{tikzcd} \qquad \begin{tikzcd} \dot{\mathbb{P}} \arrow[r, bend left, "\uparrow"] \arrow[r, phantom, "\perp"] & \mathbb{N} \arrow[l, bend left, "\downarrow"] \end{tikzcd} \tag{3}$$

For all $P \in \mathbb{P}$ and $N \in \mathbb{N}$, $\leq\!\cdot^{+} \subseteq \mathbb{P} \times \dot{\mathbb{P}}$, $\leq \subseteq \mathbb{P} \times \mathbb{N}$ and $\cdot\!\leq^{-} \subseteq \dot{\mathbb{N}} \times \mathbb{N}$ are s.t.:

$$\uparrow\! P \cdot\!\leq^{-} N \quad \text{iff} \quad P \leq N \quad \text{iff} \quad P \leq\!\cdot^{+} \downarrow\! N \tag{4}$$

i.e. $\leq$ is the **weakening relation** represented by the **heterogeneous adjunction** $\uparrow \dashv^{\leq\cdot^{+}}_{\cdot\leq^{-}} \downarrow$.

**Collage posets**: $(\mathring{\mathbb{P}}, \mathring{\lesssim}^{+}) := (\mathbb{P} \sqcup \dot{\mathbb{P}}, \leq^{+} \sqcup \leq\!\cdot^{+} \sqcup \cdot\!\leq\!\cdot^{+})$, $(\mathring{\mathbb{N}}, \mathring{\lesssim}^{-}) := (\mathbb{N} \sqcup \dot{\mathbb{N}}, \leq^{-} \sqcup \cdot\!\leq^{-} \sqcup \cdot\!\leq\!\cdot^{-})$.

**Collage weakening relation**: $\mathring{\lesssim} := \leq\!\cdot \sqcup \leq \sqcup \cdot\!\leq \subseteq \mathring{\mathbb{P}} \times \mathring{\mathbb{N}}$.

# (Heterogeneous multi-type proper) focused display calculus **fD.LG**

Notation: $\mathring{P} \in \{P, \dot{P}\}$, resp. $\mathring{N} \in \{N, \dot{N}\}$.

| | | | |
|---|---|---|---|
| $P$ | $:=$ | $p \mid \mathring{P} \otimes \mathring{P} \mid (\mathring{P} \oslash \mathring{N}) \mid (\mathring{N} \obslash \mathring{P})$ | Pure positive formulas $(\check{\downharpoonleft}\, \dot{\Delta})$ |
| $N$ | $:=$ | $n \mid (\mathring{N} \oplus \mathring{N}) \mid \mathring{P} \backslash \mathring{N} \mid \mathring{N} / \mathring{P}$ | Pure negative formulas $(\hat{\upharpoonleft}\, \dot{X})$ |
| $\dot{P}$ | $:=$ | $\downarrow N$ | Shifted positive formulas |
| $\dot{N}$ | $:=$ | $\uparrow P$ | Shifted negative formulas |

Well-formed sequents (sequents in grey cells are not derivable):

| | | | | |
|---|---|---|---|---|
| Positive sequents | $X \vdash^{+} Y$ | $\dot{X} \cdot\!\vdash^{+} Y$ | $X \vdash\!\cdot^{+} \dot{Y}$ | $\dot{X} \cdot\!\vdash\!\cdot^{+} \dot{Y}$ |
| Negative sequents | $\Delta \vdash^{-} \Gamma$ | $\dot{\Delta} \cdot\!\vdash^{-} \Gamma$ | $\Delta \vdash\!\cdot^{-} \dot{\Gamma}$ | $\dot{\Delta} \cdot\!\vdash\!\cdot^{-} \dot{\Gamma}$ |
| Neutral sequents | $X \vdash \Delta$ | $\dot{X} \cdot\!\vdash \Delta$ | $X \vdash\!\cdot\, \dot{\Delta}$ | $\dot{X} \cdot\!\vdash\!\cdot\, \dot{\Delta}$ |

(5)

Each consequence relation is interpreted by a W.R. as follows:

| $t$ | $\vdash^{+}$ | $\vdash\!\cdot^{+}$ | $\cdot\!\vdash\!\cdot^{+}$ | $\vdash^{-}$ | $\cdot\!\vdash^{-}$ | $\cdot\!\vdash\!\cdot^{-}$ | $\vdash$ | $\cdot\!\vdash$ | $\vdash\!\cdot$ | $\mathring{\vdash}^{+}$ | $\mathring{\vdash}^{-}$ | $\mathring{\vdash}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t^{\mathbb{FP}.\mathbb{LG}}$ | $\leq^{+}$ | $\leq\!\cdot^{+}$ | $\cdot\!\leq\!\cdot^{+}$ | $\leq^{-}$ | $\cdot\!\leq^{-}$ | $\cdot\!\leq\!\cdot^{-}$ | $\preceq$ | $\cdot\!\preceq$ | $\preceq\!\cdot$ | $\mathring{\leq}^{+}$ | $\mathring{\leq}^{-}$ | $\mathring{\preceq}$ |

(6)

# Axioms and cuts

$$
\frac{}{p \vdash^{+} p}\ p\text{-Id}
\qquad\qquad
n\text{-Id}\ \frac{}{n \vdash^{-} n}
$$

$$
\text{P-Cut}\ \frac{\mathring{X} \mathring{\vdash}^{+} \mathring{P} \qquad \mathring{P} \mathring{\vdash}^{+} \mathring{Y}}{\mathring{X} \mathring{\vdash}^{+} \mathring{Y}}
\qquad\qquad
\frac{\mathring{\Gamma} \mathring{\vdash}^{-} \mathring{N} \qquad \mathring{N} \mathring{\vdash}^{-} \mathring{\Delta}}{\mathring{\Gamma} \mathring{\vdash}^{-} \mathring{\Delta}}\ \text{N-Cut}
$$

$$
\text{Pn-Cut}\ \frac{\mathring{X} \mathring{\vdash}^{+} \mathring{P} \qquad \mathring{P} \mathring{\vdash} \mathring{\Delta}}{\mathring{X} \mathring{\vdash} \mathring{\Delta}}
\qquad\qquad
\frac{\mathring{X} \mathring{\vdash} \mathring{N} \qquad \mathring{N} \mathring{\vdash}^{-} \mathring{\Delta}}{\mathring{X} \mathring{\vdash} \mathring{\Delta}}\ \text{nN-Cut}
\tag{7}
$$

# Logical rules

$$
\otimes_L \ \frac{\mathring{P} \mathbin{\hat{\otimes}} \mathring{Q} \overset{\circ}{\vdash} \mathring{\Delta}}{\mathring{P} \otimes \mathring{Q} \overset{\circ}{\vdash} \mathring{\Delta}}
\qquad
\frac{\mathring{X} \overset{\circ}{\vdash}{}^{+} \mathring{P} \qquad \mathring{Y} \overset{\circ}{\vdash}{}^{+} \mathring{Q}}{\mathring{X} \mathbin{\hat{\otimes}} \mathring{Y} \vdash^{+} \mathring{P} \otimes \mathring{Q}} \ \otimes_R
$$

$$
\backslash_L \ \frac{\mathring{X} \overset{\circ}{\vdash}{}^{+} \mathring{P} \qquad \mathring{N} \overset{\circ}{\vdash}{}^{-} \mathring{\Delta}}{\mathring{P} \backslash \mathring{N} \vdash^{-} X \mathbin{\check{\backslash}} \Delta}
\qquad
\frac{\mathring{X} \overset{\circ}{\vdash} \mathring{P} \mathbin{\check{\backslash}} \mathring{N}}{\mathring{X} \overset{\circ}{\vdash} \mathring{P} \backslash \mathring{N}} \ \backslash_R
\qquad
/_L \ \frac{\mathring{N} \overset{\circ}{\vdash}{}^{-} \mathring{\Delta} \qquad \mathring{X} \overset{\circ}{\vdash}{}^{+} \mathring{P}}{\mathring{N} / \mathring{P} \vdash^{-} \mathring{\Delta} \mathbin{\check{/}} \mathring{X}}
\qquad
\frac{\mathring{X} \overset{\circ}{\vdash} \mathring{N} \mathbin{\check{/}} \mathring{P}}{\mathring{X} \overset{\circ}{\vdash} \mathring{N} / \mathring{P}} \ /_R
$$

$$
\downarrow_L \ \frac{N \vdash^{-} \Delta}{\downarrow N \cdot\!\vdash\!\cdot^{+} \check{\downarrow} \Delta}
\qquad
\frac{\mathring{X} \overset{\circ}{\vdash}{}^{+} \check{\downarrow} N}{\mathring{X} \overset{\circ}{\vdash}{}^{+} \downarrow N} \ \downarrow_R
\qquad
\uparrow_L \ \frac{\hat{\uparrow} P \overset{\circ}{\vdash}{}^{-} \mathring{\Delta}}{\uparrow P \overset{\circ}{\vdash}{}^{-} \mathring{\Delta}}
\qquad
\frac{X \vdash^{+} P}{\hat{\uparrow} X \cdot\!\vdash\!\cdot^{-} \uparrow P} \ \uparrow_R
\tag{8}
$$

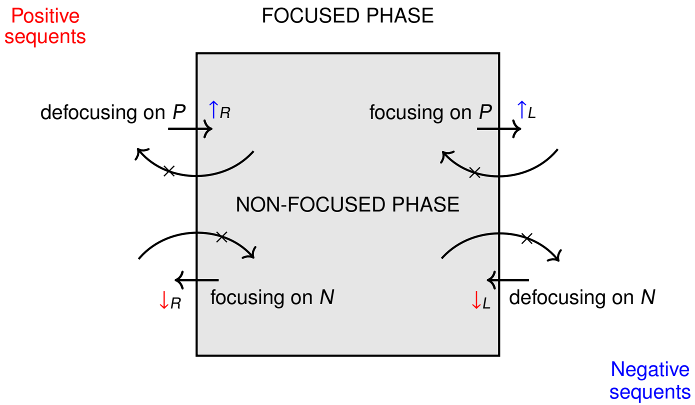
**Display postulates**:

$$
\hat{\otimes} \dashv \check{\backslash}\;
\frac{\mathring{Y} \mathrel{\mathring{\vdash}} \mathring{X} \,\check{\backslash}\, \mathring{\Delta}}
{\dfrac{\mathring{X} \,\hat{\otimes}\, \mathring{Y} \mathrel{\mathring{\vdash}} \mathring{\Delta}}
{\mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Delta} \,\check{/}\, \mathring{Y}}}
\;\hat{\otimes} \dashv \check{/}
\qquad
\frac{\mathring{X} \,\hat{\oslash}\, \mathring{\Delta} \mathrel{\mathring{\vdash}} \mathring{\Gamma}}
{\dfrac{\mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Gamma} \,\check{\oplus}\, \mathring{\Delta}}
{\mathring{\Gamma} \,\hat{\obslash}\, \mathring{X} \mathrel{\mathring{\vdash}} \mathring{\Delta}}}
\;\hat{\oslash} \dashv \check{\oplus}\;\; \hat{\obslash} \dashv \check{\oplus}
$$

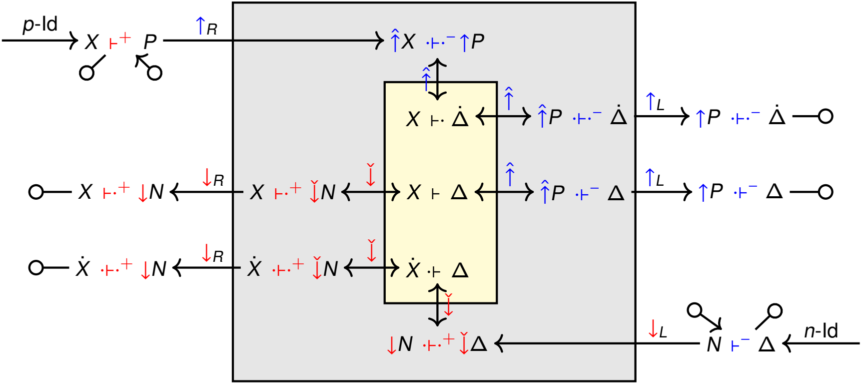
$$
\frac{\hat{\uparrow} X \mathrel{\cdot\vdash\cdot^{-}} \dot{\Delta}}{X \vdash^{+} \check{\downarrow}\dot{\Delta}}\;\hat{\uparrow} \dashv \check{\downarrow}
\qquad
\frac{\hat{\uparrow} X \mathrel{\cdot\vdash^{-}} \Delta}{X \mathrel{\vdash\cdot^{+}} \check{\downarrow}\Delta}\;\hat{\uparrow} \dashv \check{\downarrow}
\qquad
\hat{\uparrow} \dashv \check{\downarrow}\;
\frac{\dot{X} \mathrel{\cdot\vdash\cdot^{+}} \check{\downarrow}\Delta}{\hat{\uparrow}\dot{X} \vdash^{-} \Delta}
\tag{9}
$$

**Structural rules for shifts**:

$$
\frac{\mathring{X} \mathrel{\mathring{\vdash}} \Delta}{\mathring{X} \mathrel{\mathring{\vdash}^{+}} \check{\downarrow}\Delta}\;\check{\downarrow}
\qquad
\hat{\uparrow}\;\frac{X \mathrel{\mathring{\vdash}} \mathring{\Delta}}{\hat{\uparrow} X \mathrel{\mathring{\vdash}^{-}} \mathring{\Delta}}
\tag{10}
$$


Positive sequents
FOCUSED PHASE
defocusing on $P$
$\uparrow_R$
focusing on $P$
$\uparrow_L$
NON-FOCUSED PHASE
$\downarrow_R$
focusing on $N$
$\downarrow_L$
defocusing on $N$
Negative sequents

# Phases and phase transitions 2/2



$$
\dfrac{\dfrac{\dfrac{}{s_0 \vdash^- s_{11}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{np_1 \vdash^+ np_3} \quad \dfrac{}{s_4 \vdash^- s_8}}{np \backslash s \vdash^- np \,\check{\backslash}\, s} \qquad \dfrac{\dfrac{\dfrac{}{s_5 \vdash^- s_7} \quad \dfrac{}{np_9 \vdash^+ np_6}}{s / np \vdash^- s \,\check{/}\, np}}{np \vdash^{\cdot +} \downarrow(\downarrow(s / np) \backslash s)} \leftrightharpoons}{(np \backslash s) / \downarrow(\downarrow(s / np) \backslash s) \vdash^- (np \,\check{\backslash}\, s) \,\check{/}\, np}}{np \,\hat{\otimes}\, \downarrow((np \backslash s) / \downarrow(\downarrow(s / np) \backslash s)) \vdash^{\cdot +} \downarrow(s / np)} \leftrightharpoons, a \qquad \dfrac{}{s_{10} \vdash^- s_2}}{\downarrow(s_8 \backslash np_9) \backslash s_{10} \vdash^- (np \,\hat{\otimes}\, \downarrow((np \backslash s) / \downarrow(\downarrow(s / np) \backslash s))) \,\check{\backslash}\, s}}{\downarrow((np \backslash s) / \downarrow(\downarrow(s / np) \backslash s)) \,\hat{\otimes}\, \downarrow(\downarrow(s / np) \backslash s) \vdash^{\cdot +} \downarrow(np \backslash s)} \leftrightharpoons, a^{-1}}{s_0 / \downarrow(np_1 \backslash s_2) \vdash^- s \,\check{/}\, (\downarrow((np \backslash s) / \downarrow(\downarrow(s / np) \backslash s)) \,\hat{\otimes}\, \downarrow(\downarrow(s / np) \backslash s))}}{\underbrace{\downarrow(s_0 / \downarrow(np_1 \backslash s_2))}_{\textit{Everybody}} \,\hat{\otimes}\, (\underbrace{\downarrow((np_3 \backslash s_4) / \downarrow(\downarrow(s_5 / np_6) \backslash s_7))}_{\textit{needs}} \,\hat{\otimes}\, \underbrace{\downarrow(\downarrow(s_8 / np_9) \backslash s_{10})}_{\textit{somebody}}) \vdash s_{11}} \leftrightharpoons
$$

# Conclusions

**What we did.**

Lambek-Grishin logic:

- (heterogenous multi-type) focused display calculus **fD.LG** (canonical cut-elimination and strong focalization)
- fully polarized algebraic semantics $\mathbb{FP}.\mathbb{LG}$ (semantic proof of completeness of focusing)

**Future work.**

- We expect that the approach extends to every displayable logic. We conjecture that:
  - Every displayable logic **L** can be endowed with a focalized (heterogenous multi-type) display calculus **fD.L** and a fully polarized algebraic semantics $\mathbb{FP}.\mathbb{L}$ (where **fD.L** is complete w.r.t. $\mathbb{FP}.\mathbb{L}$).
  - Every focalized (heterogenous multi-type) display calculus enjoys a (i) canonical semantic proof of completeness of focusing AND a (ii) canonical syntactic proof of completeness of focusing.
- We expect that the approach can be lifted at the level of categories (using profunctors instead of weakening relations) providing a fully-fledged semantics of proofs for a given displayable logic **L**.

# Appendix A. Mutations

Let $\mathcal{C}$ be a heterogeneous multi-type calculus and let $\mathcal{Q}$ be the set of types of $\mathcal{C}$.

$\mathcal{S}_{\mathcal{F}}$ (resp. $\mathcal{S}_{\mathcal{G}}$) is the set of structural $\mathcal{F}$-connectives (resp. $\mathcal{G}$-connectives), $\mathcal{S} = \mathcal{S}_{\mathcal{F}} \cup \mathcal{S}_{\mathcal{G}}$.

$\mathcal{T}$ is the set of turnstiles.

We call **sort of** $H$, $\mathrm{sort}(H) \in \mathcal{Q}^n$, the $n$-tuple of types that the connective takes as input.

We call **sort of** $t$, $\mathrm{sort}(t) \in \mathcal{Q}^2$, the pair of types that $t$ connects.

## Definition

The **mutation** relation of $\mathcal{C}$, $\mu_{\mathcal{C}} \subseteq \mathcal{S} \times \mathcal{S}$, is an equivalence relation between structural connectives s.t.:

1. if $H\mu_{\mathcal{C}}H'$ then $H \in \mathcal{S}_{\mathcal{F}}$ if and only if $H' \in \mathcal{S}_{\mathcal{F}}$;
2. if $H\mu_{\mathcal{C}}H'$ then $H$ and $H'$ have the same arity;
3. if $H\mu_{\mathcal{C}}H'$ and $\mathrm{sort}(H) = \mathrm{sort}(H')$ then $H = H'$.

Informally, the mutation relation describes into which structural connectives the structural connective $H$ can be mutated.

We can extend the relation to (not necessarily well typed) structures recursively on the generation tree of a structure: We say that $\Phi\mu\Psi$ if the generation trees of $\Phi$ and $\Psi$ are identical modulo $\mu$.

By condition 3 of Definition 4, given a structure $\Phi$, there exists at most one well typed structure in $\mu[\Phi] = \{\Psi \mid \Phi\mu\Psi\}$, which we denote with $\mu(\Phi)$.

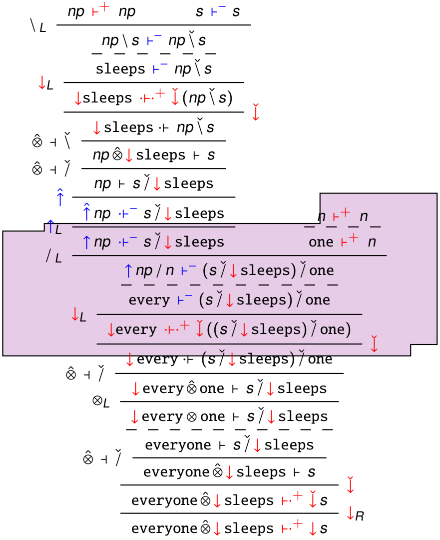
$$
\dfrac{\begin{array}{c}\pi_1\\ p\,\hat{\otimes}\,(\downarrow p\backslash n)\ \vdash\!\cdot^{+}\ \downarrow n\end{array}\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_2\\ \downarrow n\ \cdot\!\vdash\!\cdot^{+}\ \check{\downarrow} n\\ \vdots\\ \downarrow n\,\hat{\otimes}\,p\ \vdash\!\cdot\ \uparrow(\downarrow n\otimes p)\end{array}}{\downarrow n\ \cdot\!\vdash\ \uparrow(\downarrow n\otimes p)\,\check{/}\,p}\ \hat{\otimes}\dashv\check{/}}{\downarrow n\ \cdot\!\vdash\!\cdot^{+}\ \check{\downarrow}(\uparrow(\downarrow n\otimes p)\,\check{/}\,p)}\ \check{\downarrow}}{\hat{\uparrow}\downarrow n\ \vdash^{-}\ \uparrow(\downarrow n\otimes p)\,\check{/}\,p}\ \hat{\uparrow}\dashv\check{\downarrow}}{\downarrow n\ \cdot\!\vdash\!\cdot^{+}\ \check{\downarrow}(\uparrow(\downarrow n\otimes p)\,\check{/}\,p)}\ \hat{\uparrow}\dashv\check{\downarrow}}{\downarrow n\ \cdot\!\vdash\ \uparrow(\downarrow n\otimes p)\,\check{/}\,p}\ \check{\downarrow}}{p\,\hat{\otimes}\,(\downarrow p\backslash n)\ \vdash\ \uparrow(\downarrow n\otimes p)\,\check{/}\,p}\ \text{P-Cut}
$$

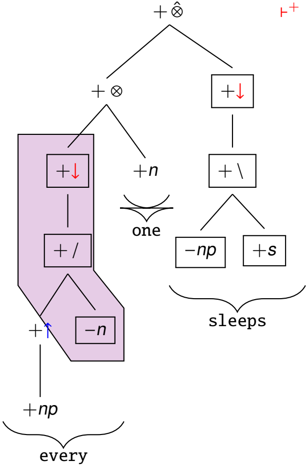
$$\rightsquigarrow$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\pi_1\\ p\,\hat{\otimes}\,(\downarrow p\backslash n)\ \vdash\!\cdot^{+}\ \downarrow n\end{array}\qquad\begin{array}{c}\pi_2\\ \downarrow n\ \cdot\!\vdash\!\cdot^{+}\ \check{\downarrow} n\end{array}}{p\,\hat{\otimes}\,(\downarrow p\backslash n)\ \vdash\!\cdot^{+}\ \check{\downarrow} n}\ \text{P-Cut}\\ \vdots\\ (p\,\hat{\otimes}\,(\downarrow p\backslash n))\,\hat{\otimes}\,p\ \vdash\!\cdot\ \uparrow(\downarrow n\otimes p)\end{array}}{p\,\hat{\otimes}\,(\downarrow p\backslash n)\ \vdash\ \uparrow(\downarrow n\otimes p)\,\check{/}\,p}\ \hat{\otimes}\dashv\check{/}}{p\,\hat{\otimes}\,(\downarrow p\backslash n)\ \vdash\!\cdot^{+}\ \check{\downarrow}(\uparrow(\downarrow n\otimes p)\,\check{/}\,p)}\ \check{\downarrow}}{\hat{\uparrow}(p\,\hat{\otimes}\,(\downarrow p\backslash n))\ \cdot\!\vdash^{-}\ \uparrow(\downarrow n\otimes p)\,\check{/}\,p}\ \hat{\uparrow}\dashv\check{\downarrow}}{p\,\hat{\otimes}\,(\downarrow p\backslash n)\ \vdash\!\cdot^{+}\ \check{\downarrow}(\uparrow(\downarrow n\otimes p)\,\check{/}\,p)}\ \hat{\uparrow}\dashv\check{\downarrow}}{p\,\hat{\otimes}\,(\downarrow p\backslash n)\ \vdash\ \uparrow(\downarrow n\otimes p)\,\check{/}\,p}\ \check{\downarrow}
$$

# Appendix B. Focused phases and maximal PIA subtrees

The purple area is the proof-section including all the tonicity rules used to build the PIA subtree of `everyone`:



# References


[And92] Jean-Marc Andreoli.
Logic programming with focusing proofs in linear logic.
*Journal of Logic and Computation*, 2(3):297–347, 1992.

[And01] Jean-Marc Andreoli.
Focussing and proof construction.
*Annals of Pure and Applied Logic*, 107(1):131 – 163, 2001.

[Bas12] Arno Bastenhof.
Polarized Montagovian semantics for the Lambek-Grishin calculus.
In P. de Groote and MJ. Nederhof, editors, *Formal Grammar*, volume 7395 of *Lecture Notes in Computer Science*. Springer, Berlin, Heidelberg, 2012.

[Ben73] Jean Benabou.
*Les distributeurs: d'après le cours de questions spéciales de mathématique*.
Rapport n. 33 du Séminaire de Mathématique Pure. Institut de mathématique pure et appliquée, Université Catholique de Louvain, 1973.

[GJL+18] Giuseppe Greco, Peter Jipsen, Fei Liang, Alessandra Palmigiano, and Apostolos Tzimoulis.
Algebraic proof theory for LE-logics, 2018.

[Mil04] Dale Miller.
*An Overview of Linear Logic Programming*, page 119–150.
London Mathematical Society Lecture Note Series. Cambridge University Press, 2004.

[MM12] Michael Moortgat and Richard Moot.
Proof nets and the categorial flow of information.
In A. Baltag, D. Grossi, A. Marcoci, B. Rodenhäuser, and S. Smets, editors, *Logic and Interactive RAtionality. Yearbook 2011*, pages 270–302. ILLC, University of Amsterdam, 2012.

[Moo09] Michael Moortgat.
Symmetric categorial grammar.
*Journal of Philosophical Logic*, 38(6):681–710, 2009.