Quantum logics as algebras for monads

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The plan

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- Effect algebras and other quantum logics

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  - Pseudo-effect algebras over **BPos**
  - ω-effect algebras over **BPos**
  - Orthomodular posets over **BPosInv**

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- Effect algebras are algebras for the Kalmbach monad on **BPos**
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  - ω-effect algebras over **BPos**
  - Orthomodular posets over **BPosInv**
- Future research

Effect Algebras

Foulis and Bennett 1994; Kôpka and Chovanec 1994; Giuntini and Greuling 1989

An effect algebra is a partial algebra $(E; +, 0, 1)$ satisfying the following conditions.

(E1) If $a + b$ is defined, then $b + a$ is defined and $a + b = b + a$.

(E2) If $a + b$ and $(a + b) + c$ are defined, then $b + c$ and $a + (b + c)$ are defined and $(a + b) + c = a + (b + c)$.

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a + a' = 1$.

(E4) If $a + 1$ exists, then $a = 0$

Basic Relationships

Let E be an effect algebra.

- Cancellativity: $a + b = a + c \Rightarrow b = c$.
- Partial difference: If $a + b = c$ then we write $a = c - b$. The operation $-$ is well defined and $a' = 1 - a$.
- Poset: Write $b \leq c$ iff $\exists a : a + b = c$; $(E, \leq)$ is then a bounded poset.

Classes of Effect Algebras

The class of effect algebras includes

- modular ortholattices (Birkhoff and Von Neumann, 1936)
- orthomodular lattices (Husimi, 1937)
- orthomodular posets (Finch, 1970)
- orthoalgebras (Foulis and Randall, 1981)
- MV-algebras (Chang, 1959)
- Any interval $[0, u]$ in the positive cone of an abelian po-group.
- Boolean algebras.

D-posets

A <u>D-poset</u> is a system $(P; \leq, -, 0, 1)$ consisting of a partially ordered set P bounded by 0 and 1 with a partial binary operation – satisfying the following conditions.

(D1) $b - a$ is defined if and only if $a \leq b$.

(D2) If $a \leq b$, then $b - a \leq b$ and $b - (b - a) = a$.

(D3) If $a \leq b \leq c$, then $c - b \leq c - a$ and $(c - a) - (c - b) = b - a$.

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Every D-poset is an effect algebra and vice versa.

$$a + b = 1 - \big((1 - a) - b\big)$$

Orthomodular posets

(Finch, 1970)

An orthomodular poset is a bounded poset with involution $(A, \leq, ', 0, 1)$ satisfying the following conditions, for all $x, y \in A$.

- $x \wedge x' = 0$.
- If $x \leq y'$, then $x \vee y$ exists.
- If $x \leq y$, then $x \vee (x \vee y')' = y$.

An orthomodular lattice is an orthomodular poset that is a lattice.

The Kalmbach construction
(Kalmbach, 1977; Mayet and Navara, 1995)

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- Introduce a partial order on the set $K(A)$ by the following rule:

$$[a_1 < a_2 < \cdots < a_{2n-1} < a_{2n}] \leq [b_1 < b_2 < \cdots < b_{2n-1} < b_{2k}]$$

if and only if for every $i \in \{1, \dots, n\}$ there exists $j \in \{1, \dots, n\}$ such that $b_{2j-1} \leq a_{2i-1} < a_{2i} \leq b_{2j}$.

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- Equip $K(A)$ with the unary operation $C \mapsto C'$ given by the rule

$$C' = C \Delta \{0, 1\}$$

  where Δ is the symmetric difference.

The Kalmbach construction

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$$\eta_A(a) = \begin{cases} [0 < a] & \text{if } 0 < a \\ \emptyset & \text{if } a = 0 \end{cases}$$

is an embedding of A into $K(A)$.

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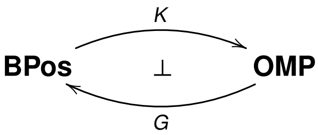
Corollary

Every bounded lattice is a bounded sublattice of an orthomodular lattice.

Where does the Kalmbach construction come from

Theorem

Harding (2004) K is a functor left adjoint to the forgetful functor G from the category of orthomodular posets to the category of bounded posets.



However, K does not restrict to a functor from bounded lattices (with lattice homomorphisms) to orthomodular lattices.

Adjunctions and monads

- Every adjunction induces a monad on the domain category of the left adjoint functor.
- Every monad T on a category $\mathcal{C}$ gives rise to a category of algebras $\mathcal{C}^T$ and an adjunction

$$\mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{C}^T$$

- For every adjunction

$$\mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{D}$$

  that induces T there is a canonical comparison functor $\mathcal{D} \to \mathcal{C}^T$.
- An adjunction is <u>monadic</u> if the comparison functor is an equivalence of categories.

Is this adjunction monadic?

$$\mathbf{BPos} \underset{U}{\overset{K}{\rightleftarrows}} \mathbf{OMP} \qquad (K \dashv U)$$

Is this adjunction monadic?

$$\mathbf{BPos} \underset{U}{\overset{K}{\underset{\bot}{\rightleftarrows}}} \mathbf{OMP}$$

Answer

No.

The category of algebras for the Kalmbach monad

We say that the monad on **BPos** induced by the adjunction between **BPos** and **OMP** is the Kalmbach monad

Theorem

(Jenča, 2015) The category of algebras for the Kalmbach monad is equivalent to the category of effect algebras **EA**.

Pseudo effect algebras

Definition

Dvurečenskij and Vetterlein (2001) A pseudo effect algebra is an algebra A with a partial binary operation $+$ and two constants $0, 1$ such that, for all $a, b, c \in A$.

- (PE1) If $a + (b + c)$ exists, then $(a + b) + c$ exists and $a + (b + c) = (a + b) + c$.
- (PE2) There is exactly one d and exactly one e such that $a + d = e + a = 1$.
- (PE3) If $a + b$ exists, there are d, e such that $d + a = b + e = a + b$.
- (PE4) If $a + 1$ exists or $1 + a$ exists, then $a = 0$.

Pseudo effect algebras are algebras for a monad on **BPos**

Theorem

(Jenča, 2020) The forgetful functor from the category of pseudo effect algebras to the category of bounded posets is a right adjoint functor of a monadic adjunction.

ω-effect algebras

Definition

We say that an effect algebra E is ω-effect algebra when every increasing sequence $a_1 \leq a_2 \leq \cdots$ in E has a supremum. A morphism of ω-effect algebras is a morphism of effect algebras that preserves suprema of increasing sequences.

ω-effect algebras are algebras for a monad on **BPos**

Theorem

(van de Wetering, 2021) The forgetful functor from the category ω-effect algebras to the category of bounded posets is a right adjoint functor of a monadic adjunction.

What about orthomodular posets?

- Recall, that there is an adjunction between **BPos** and **OMP**.
- However, this adjunction is non-monadic.
- Can we represent orthomodular posets as algebras for a monad?

Orthomodular posets are algebras for a monad on **BPosInv**

Theorem

(Jenča, 2022) The forgetful functor from the category **OMP** *to the category of bounded posets with involution is a right adjoint functor of a monadic adjunction.*

Conclusion, future research

There are monadic adjunctions

$$\mathbf{BPos} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{PsEA}$$

$$\mathbf{BPos} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \omega\mathbf{EA}$$

$$\mathbf{BPosInv} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{OMP}$$

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The proof of all these uses

- General adjoint functor theorem
- Beck's monadicity theorem

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Problem

Give an explicit description of the left adjoint functor in these adjunctions.

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