Presenting quotient locales

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Frame presentations

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For example, the frame of reals may be presented as

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\begin{aligned}
\mathcal{O}\mathbb{R} = \Big\langle (\!(p,q)\!) \text{ for } p,q \in \mathbb{Q} \sqcup \{-\infty,\infty\} \;\Big|\; & (\!(-\infty,\infty)\!) = 1, \\
& (\!(p,q)\!) \wedge (\!(p',q')\!) = (\!(\max(p,p'),\min(q,q'))\!), \\
& (\!(p,q)\!) \vee (\!(p',q')\!) = (\!(p,q')\!) \text{ for } p \leq p' < q \leq q', \\
& (\!(p,q)\!) = \bigvee\nolimits_{p<p'<q'<q} (\!(p',q')\!) \Big\rangle.
\end{aligned}
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Sublocales, quotient frames, quotients locales and subframes

Sublocales $Y \hookrightarrow X$ are the pointfree analogue of subspaces and correspond to *quotient* frames $\mathcal{O}X \twoheadrightarrow \mathcal{O}Y$.

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Quotient locales $X \twoheadrightarrow Y$ correspond to certain *subframes* $\mathcal{O}Y \hookrightarrow \mathcal{O}X$.

We would like to obtain a presentation of $\mathcal{O}Y$ from one for $\mathcal{O}X$.

Open quotients of locales

It will be helpful to restrict to important subclasses of quotients.

Definition

A locale map $f \colon X \to Y$ is open if its corresponding frame map $f^* \colon \mathcal{O}Y \to \mathcal{O}X$ has a left adjoint $f_! \colon \mathcal{O}X \to \mathcal{O}Y$ and these satisfy $f_!(a \wedge f^*(b)) = f_!(a) \wedge b$.

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Note that since $q_!$ is a left adjoint, it preserves all joins.

We call a poset admitting all joins a suplattice and write **Sup** for the category of suplattices and join-preserving maps.

Open quotients and closure operators

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Open quotients of X correspond to join-preserving closure operators $j\colon \mathcal{O}X \to \mathcal{O}X$ satisfying $j(a) \wedge j(b) = j(a \wedge j(b))$.

Proposition

Suppose $f, g\colon R \to X$ are open locale maps. Then their coequaliser is an open quotient.

$$\begin{tikzcd} R \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \arrow[r, two heads, "e"] & Y \end{tikzcd}$$

Moreover, the associated closure operator is given by

$$\bigvee_{n=0}^{\infty} (f_! g^*)^n \vee \bigvee_{n=0}^{\infty} (g_! f^*)^n.$$

Quotient presentations

We can now state our problem more formally.

- Suppose $\mathcal{O}X$ is given by a presentation $\langle G \mid R \rangle$.
- Let j be an 'open' closure operator on $\mathcal{O}X$ and let $\mathcal{O}Y \hookrightarrow \mathcal{O}X$ be its frame of fixed points.
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If we could relate suplattice and frame presentations, we could proceed in a similar way to as with a frame quotient.

The suplattice coverage theorem

Definition

Let G be a $\wedge$-semilattice. We will call $\langle G\ \wedge\text{-semilattice} \mid R\rangle_{\mathbf{Frm}}$ a **Sup**-type frame presentation if every relation in R is of the form $\bigvee A \leq \bigvee B$ and furthermore, if $\bigvee A \leq \bigvee B$ is a relation, then so is $\bigvee_{a \in A} a \wedge c \leq \bigvee_{b \in B} b \wedge c$ for each $c \in G$.

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Theorem (Johnstone 1982, Abramsky and Vickers 1993)

For a **Sup**-*type frame presentation given by G and R, there is an order isomorphism*

$$\langle G\ \wedge\text{-semilattice} \mid R\rangle_{\mathbf{Frm}} \cong \langle G\ \text{poset} \mid R\rangle_{\mathbf{Sup}}.$$

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Proposition

Suppose $\mathcal{O}X = \langle G \ \wedge\text{-semilattice} \mid R \rangle_{\mathsf{Frm}}$ is a Sup*-type presentation and let $q \colon X \to Y$ be an open quotient. Then*

$$\mathcal{O}Y \cong \langle \Diamond g \ \text{ for } g \in G \mid R,\ \Diamond 1 = 1, \\ \Diamond s \wedge \Diamond t = \Diamond(s \wedge q^* q_!(t)),\ s, t \in G \rangle_{\mathsf{Frm}},$$

where we interpret $\Diamond(s \wedge q^* q_!(t)) = \bigvee_\beta \Diamond(s \wedge t_\beta)$ for specified representation $q^* q_!(t) = \bigvee_\beta t_\beta$.

The circle via $\mathbb{R} \twoheadrightarrow \mathbb{T}$

Recall our presentation for $\mathcal{O}\mathbb{R}$ from before.

$$\begin{aligned}
\mathcal{O}\mathbb{R} = \Big\langle (\!(p,q)\!) \text{ for } p,q \in \mathbb{Q} \sqcup \{-\infty,\infty\} \;\Big|\; & \\
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Let us find a presentation for the circle $\mathbb{T}$ from the coequaliser

$$\mathbb{R} \underset{+1}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbb{R} \twoheadrightarrow \mathbb{T}.$$

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The corresponding closure operator is

$$\bigvee_{n\in\mathbb{N}} (\mathrm{id}_! \circ (+1)^*)^n \vee \bigvee_{n\in\mathbb{N}} ((+1)_! \circ \mathrm{id}^*)^n$$

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This sends $(\!(p, q)\!)$ to $\bigvee_{n\in\mathbb{Z}} (\!(p-n, q-n)\!)$.

The circle via $\mathbb{R} \twoheadrightarrow \mathbb{T}$

We then have the same generators for $\mathcal{O}\mathbb{T}$ as for $\mathcal{O}\mathbb{R}$: $(\!(p, q)\!)$ for $p, q \in \mathbb{Q} \sqcup \{-\infty, \infty\}$.

The relations become:

- $(\!(-\infty, \infty)\!) = 1$,
- $(\!(p, q)\!) \wedge (\!(p', q')\!) = \bigvee_{n\in\mathbb{Z}} (\!(\max(p, p' - n), \min(q, q' - n))\!)$,
- $(\!(p, q)\!) \vee (\!(p', q')\!) = (\!(p, q')\!)$ for $p \leq p' < q \leq q'$,
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A similar result can be proved for proper quotients. In this case, the map q^*q_* is an *interior operator* that is also a *preframe* homomorphism.

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This time we make use of the preframe coverage theorem and the resulting presentations involve modifying the finite *joins* in a '**PreFrm**-type' frame presentation.

In fact, the result can even be generalised to triquotients by replacing suplattices and preframes by dcpos.

An example of a proper quotient

An appropriate presentation for $\mathcal{O}[0,1]$ is given by

$$\begin{aligned}\mathcal{O}[0,1] = \langle \rrparenthesis p, q \llparenthesis \ \text{ for } p, q \in \mathbb{Q} \cap [0,1] \ \big| \\ &\rrparenthesis 0, 1 \llparenthesis = 0, \\ &\rrparenthesis p, q \llparenthesis \vee \rrparenthesis p', q' \llparenthesis = \rrparenthesis \max(p, p'), \min(q, q') \llparenthesis, \\ &\rrparenthesis p, q \llparenthesis \wedge \rrparenthesis p', q' \llparenthesis = \rrparenthesis p, q' \llparenthesis \ \text{ for } p \le p' \le q \le q', \\ &\rrparenthesis p, q \llparenthesis = 1 \ \text{ for } p > q, \\ &\rrparenthesis p, q \llparenthesis = \textstyle\bigvee^{\uparrow}_{p' < p \le q < q'} \rrparenthesis p', q' \llparenthesis \ \text{ for } p > 0, q < 1 \rangle.\end{aligned}$$

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The resulting interior operators acts on generators as follows.

$$\rrparenthesis p, q \llparenthesis \mapsto \begin{cases} \rrparenthesis p, q \llparenthesis \wedge \rrparenthesis 0, 0 \llparenthesis \wedge \rrparenthesis 1, 1 \llparenthesis & \text{if } p = 0 \text{ or } q = 1 \\ \rrparenthesis p, q \llparenthesis & \text{otherwise} \end{cases}$$

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& \rrparenthesis p, q \llparenthesis \vee \rrparenthesis 0, q' \llparenthesis = \rrparenthesis p, \min(q, q') \llparenthesis \wedge \rrparenthesis 1, q \llparenthesis, \\
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