Transfer theorems for finitely subdirectly irreducible algebras

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This Talk

It is well known that . . .

Under certain conditions, some algebraic properties lift from the class of **subdirectly irreducibles** of a variety to the whole variety.

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We show that . . .

Under certain (weaker) conditions, some algebraic properties lift from the class of **finitely subdirectly irreducibles** of a variety to the whole variety.

Subdirectly Irreducible Algebras

An algebra **A** is **subdirectly irreducible** if whenever **A** is isomorphic to a subdirect product of a set of algebras, it is isomorphic to one of them; equivalently, Δ_A is completely meet-irreducible in $\mathrm{Con}\,\mathbf{A}$.

Fix a variety $\mathcal{V}$ and let $\mathcal{V}_{\mathsf{FSI}}$ and $\mathcal{V}_{\mathsf{SI}}$ denote the classes of finitely subdirectly irreducible and subdirectly irreducible members of $\mathcal{V}$, respectively.

Remark

If $\mathcal{V}$ has equationally definable principal meets, $\mathcal{V}_{\mathsf{FSI}}$ is a **universal class**. For example, if $\mathcal{V}$ is a variety of semilinear residuated lattices, $\mathcal{V}_{\mathsf{FSI}}$ is the class of totally ordered members of $\mathcal{V}$.

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An algebra **A** is **finitely subdirectly irreducible** if whenever **A** is isomorphic to a subdirect product of a *non-empty finite* set of algebras, it is isomorphic to one of them; equivalently, Δ_A is meet-irreducible in $\mathrm{Con}\,\mathbf{A}$.

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The Congruence Extension Property

A class $\mathcal{K}$ has the **congruence extension property** (CEP) if for any $\mathbf{B} \in \mathcal{K}$, subalgebra $\mathbf{A}$ of $\mathbf{B}$, and $\Theta \in \mathrm{Con}\,\mathbf{A}$, we have $\mathrm{Cg}_{\mathbf{B}}(\Theta) \cap A^2 = \Theta$.

Theorem (Davey 1977)

Let $\mathcal{V}$ be a congruence-distributive variety such that $\mathcal{V}_{\mathsf{SI}}$ is elementary. Then $\mathcal{V}$ has the CEP if and only if $\mathcal{V}_{\mathsf{SI}}$ has the CEP.

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Lifting the CEP from FSIs

Theorem

Let $\mathcal{V}$ be a congruence-distributive variety. Then $\mathcal{V}$ has the CEP if and only if $\mathcal{V}_{\mathsf{FSI}}$ has the CEP.

Proof sketch. Suppose for the non-trivial direction that $\mathcal{V}_{\mathsf{FSI}}$ has the CEP and consider any $\mathbf{B} \in \mathcal{V}$, subalgebra $\mathbf{A}$ of $\mathbf{B}$, and $\Theta \in \mathrm{Con}\,\mathbf{A}$. Assume towards a contradiction that there is some $\langle a, b\rangle \in \mathrm{Cg}_{\mathbf{B}}(\Theta) \cap A^2$ not in Θ. Zorn's Lemma yields a $\Psi^* \in \mathrm{Con}\,\mathbf{B}$ maximal w.r.t. $\langle a, b\rangle \notin (\Psi \cap A^2) \vee \Theta$, and it follows easily that Ψ^* is meet-irreducible and $\mathbf{B}/\Psi^* \in \mathcal{V}_{\mathsf{FSI}}$.

We show that for $\Phi := ((\Psi^* \cap A^2) \vee \Theta)/(\Psi^* \cap A^2) \in \mathrm{Con}\,\mathbf{A}/(\Psi^* \cap A^2)$,

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Hence we have a congruence on a subalgebra of the finitely subdirectly irreducible algebra $\mathbf{B}/\Psi^*$ that does not extend to a congruence on $\mathbf{B}/\Psi^*$, contradicting our assumption that $\mathcal{V}_{\mathsf{FSI}}$ has the CEP. □

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A class $\mathcal{K}$ has the **extension property** (EP) if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, embedding $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, and surjective homomorphism $\varphi_C \colon \mathbf{A} \to \mathbf{C}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$.

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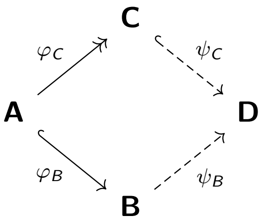

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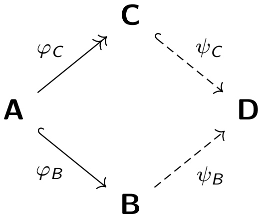

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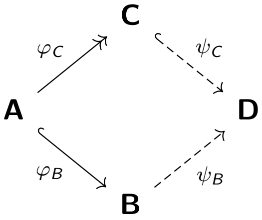

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The CEP and EP

Theorem

Let $\mathcal{V}$ be a congruence-distributive variety such that $\mathcal{V}_{\text{FSI}}$ is closed under subalgebras. Then the following are equivalent:

(1) $\mathcal{V}$ has the CEP.

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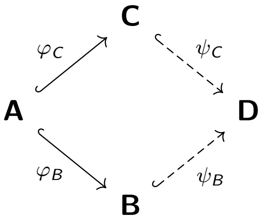

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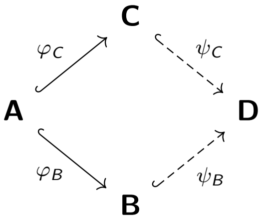
A **V-formation** in a class $\mathcal{K}$ consists of some $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, $\varphi_C \colon \mathbf{A} \to \mathbf{C}$.

An **amalgam** of this V-formation in a class $\mathcal{K}'$ consists of some $\mathbf{D} \in \mathcal{K}'$ and embeddings $\psi_B \colon \mathbf{B} \to \mathbf{D}$, $\psi_C \colon \mathbf{C} \to \mathbf{D}$ satisfying $\psi_B \varphi_B = \psi_C \varphi_C$.



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Let $\mathcal{V}_{\mathsf{SI}}^{+}$ denote the class of trivial or subdirectly irreducible algebras of $\mathcal{V}$.

Theorem (Grätzer and Lakser 1971)

Suppose that $\mathcal{V}$ has the CEP and $\mathcal{V}_{\mathsf{SI}}^{+}$ is closed under subalgebras. Then $\mathcal{V}$ has the AP if and only if every V-formation in $\mathcal{V}_{\mathsf{SI}}^{+}$ has an amalgam in $\mathcal{V}$.

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Proposition (Metcalfe, Montagna, and Tsinakis 2014)

Let $\mathcal{S}$ be a subclass of $\mathcal{V}$ satisfying

(i) $\mathcal{V}_{\mathsf{SI}} \subseteq \mathcal{S}$;

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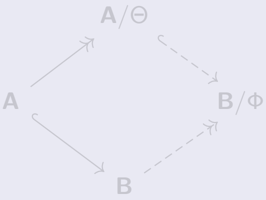

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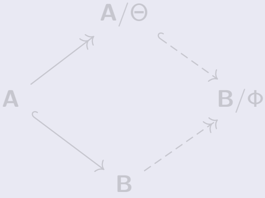

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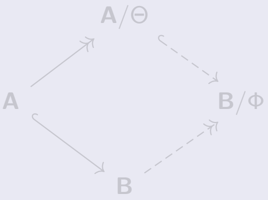

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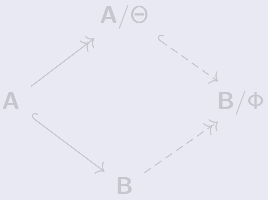

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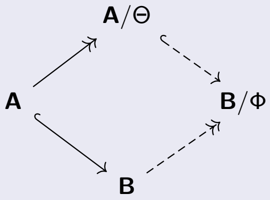

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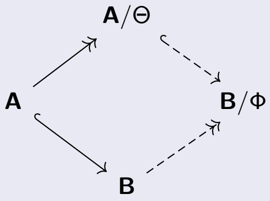

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Consider a subalgebra $\mathbf{A}$ of $\mathbf{B} \in \mathcal{V}$ and some meet-irreducible $\Theta \in \mathrm{Con}\,\mathbf{A}$. We need to show that $\Phi \cap A^2 = \Theta$ for some meet-irreducible $\Phi \in \mathrm{Con}\,\mathbf{B}$. By the CEP, $\mathrm{Cg}_{\mathbf{B}}(\Theta) \cap A^2 = \Theta$, so $T := \{\Psi \in \mathrm{Con}\,\mathbf{B} \mid \Psi \cap A^2 = \Theta\} \neq \emptyset$ and, by Zorn's Lemma, $\langle T, \subseteq \rangle$ has a maximal element Φ.

Now consider $\Phi = \Phi_1 \cap \Phi_2$ with $\Phi_1, \Phi_2 \in \mathrm{Con}\,\mathbf{B}$. Then

$$(\Phi_1 \cap A^2) \cap (\Phi_2 \cap A^2) = \Phi_1 \cap \Phi_2 \cap A^2 = \Phi \cap A^2 = \Theta$$

and, since Θ is meet-irreducible, $\Phi_1 \cap A^2 = \Theta$ or $\Phi_2 \cap A^2 = \Theta$. So $\Phi_1 \in T$ or $\Phi_2 \in T$. Hence, by maximality, $\Phi_1 = \Phi$ or $\Phi_2 = \Phi$. □

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Is there a Converse?

It is not the case in general that when $\mathcal{V}$ has the AP, also $\mathcal{V}_{\text{FSI}}$ has the AP.

The variety $\mathcal{DL}$ of distributive lattices is congruence-distributive and has the CEP and AP. Up to isomorphism, $\mathcal{DL}_{\text{FSI}}$ contains just a trivial lattice $\mathbf{1}$ and two-element distributive lattice $\mathbf{2}$.

However, $\mathbf{1}$ embeds into $\mathbf{2}$ in two different ways, giving a V-formation in $\mathcal{DL}_{\text{FSI}}$ that clearly has no amalgam in $\mathcal{DL}_{\text{FSI}}$.

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One-Sided Amalgamation Property

We say that a class $\mathcal{K}$ has the **one-sided amalgamation property** (1AP) if for any V-formation with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, $\varphi_C \colon \mathbf{A} \to \mathbf{C}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B \varphi_B = \psi_C \varphi_C$.

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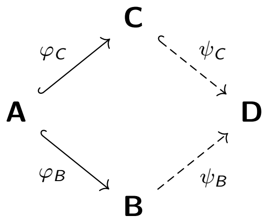

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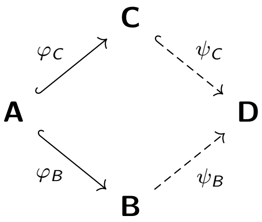

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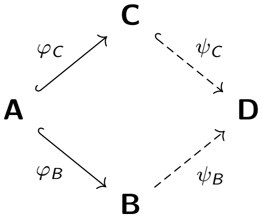

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Lifting the 1AP from FSIs

Theorem

Let $\mathcal{V}$ be a variety with the congruence extension property such that $\mathcal{V}_{\text{FSI}}$ is closed under subalgebras. Then the following are equivalent:

(1) $\mathcal{V}$ has the AP.

(2) $\mathcal{V}$ has the 1AP.

(3) $\mathcal{V}_{\text{FSI}}$ has the 1AP.

(4) Every V-formation in $\mathcal{V}_{\text{FSI}}$ has an amalgam in $\mathcal{V}_{\text{FSI}} \times \mathcal{V}_{\text{FSI}}$.

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Deciding the CEP and AP

Let $\mathcal{V}$ be a congruence-distributive variety generated by a given finite set of finite algebras of finite signature with $\mathcal{V}_{\mathsf{FSI}}$ closed under subalgebras.

(i) By Jónsson's Lemma, a finite set $\mathcal{V}^*_{\mathsf{FSI}} \subseteq \mathcal{V}_{\mathsf{FSI}}$ of finite algebras can be constructed such that each $\mathbf{A} \in \mathcal{V}_{\mathsf{FSI}}$ is isomorphic to some $\mathbf{A}^* \in \mathcal{V}^*_{\mathsf{FSI}}$.

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Concluding Remarks

- Under certain conditions, some properties transfer from the finitely subdirectly irreducibles of a variety to the whole variety, and in some cases back again.
- These include the congruence extension property, amalgamation property, transferable injections property, and also having surjective epimorphisms (Campercholi 2018). *Is there a more general approach?*
- Our results are very useful for studying semilinear residuated lattices; e.g., we have classified the varieties generated by classes of "one-component" totally ordered BL-algebras that have the amalgamation property. *Can they useful be in other contexts?*

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