Associativity in Quantum Logic

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Birkhoff/von Neumann Approach to Quantum Logic

quantum events/properties $\Longleftrightarrow$ projection operators on $\mathcal{H}$, a complex separable Hilbert space

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- $\Pi(\mathcal{H}) := \{P_X : v \mapsto v_X\}$ is the set of projection operators
- $\neg P_X := P_{X^\perp}$
- $P_X \wedge P_Y := P_{X \cap Y}$
- $P_X \vee P_Y := P_{(X \cup Y)^{\perp\perp}}$
- $0 := P_{\{\mathbf{0}\}}$
- $1 := P_H$

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quantum events/properties $\iff$ projection operators on $\mathcal{H}$, a complex separable Hilbert space

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$(\Pi(\mathcal{H}), \wedge, \vee, \neg, 0, 1)$ is an example of an *orthomodular lattice*

See [Birkhoff and von Neumann, 1936].

OL and OML

Definition

An *involutive lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \neg)$ where:

- $(A, \wedge, \vee)$ is a lattice
- $\neg$ is an antitone involution on $\mathbf{A}$

$\mathbf{A}$ is called *bounded* if it has a *bottom* 0 and *top* 1.

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An *ortholattice* is a bounded involutive lattice $\mathbf{A} = (A, \wedge, \vee, \neg, 0, 1)$ where $\neg$ is an ortho-complementation, i.e., $x \wedge \neg x \approx 0$.

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Definition

An *orthomodular lattice* (OML) is an ortholattice satisfying:

(orthomodular law) $\quad x \leq y \implies y \approx x \vee (\neg x \wedge y)$

Ortholattices form a variety OL and OMLs form a variety OML.

The problems

- It is unknown whether OML admit any form of completions
  - Not closed under MacNeille completions [Harding, 1991]
  - Not closed under canonical completions [Harding, 1998]
- The decidability of OML remains unknown
- No pair of operations form a (two-sided) residuated pair (see [Dalla Chiara et al., 2004])

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New approaches: Zooming out

- Sasaki operations form a one-sided residuated pair
- Orthomodular groupoids (OG) [Chajda and Länger, 2017]
- Pointed left-residuated ℓ-groupoids (PLRG) [Fazio et al., 2021]
- Sequent calculus for OG and PLRG [S. et al., 2022]
- Residuated ortholattices [Fussner and S., 2021]

Sasaki operations [Sasaki, 1954] are definable in involutive lattices:

$$\begin{aligned} x \cdot y &:= x \wedge (\neg x \vee y) && \text{(Sasaki product)} \\ x \to y &:= \neg x \vee (x \wedge y) && \text{(Sasaki hook)} \end{aligned}$$

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We say the triple (a, b, c) *associates in* $\mathbf{A}$, denoted $A(a, b, c)$, if

$$ab \cdot c = a \cdot bc.$$

Lemma

Let $\mathbf{A}$ *be an involutive lattice. Then:*

(1) *Sasaki product* $\cdot$ *is alternative, i.e.,* $A(x, x, y)$ *and* $A(x, y, y)$

(2) $(xy)x \approx xy$.

(3) *If* $\mathbf{A}$ *satisfies the identity* $x(y \vee z) \approx xy \vee xz$, *then* $\cdot$ *is flexible, i.e.,* $A(x, y, x)$.

(4) *If* $\mathbf{A}$ *is flexible then it satisfies* $(xy)(yx) = xy$.

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Proposition

Let $\mathbf{A}$ be a an ortholattice. Then the following are equivalent:

1. $\mathbf{A}$ is an OML.
2. $\mathbf{A} \vDash x \leq y \implies y \approx \neg x \to y$
3. $\mathbf{A} \vDash x \leq y \implies x \approx y \cdot x$.

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4. $(\cdot, \to)$ form a *(right-) residuated pair:* $\mathbf{A} \vDash x \cdot y \leq z \Leftrightarrow y \leq x \to z$.

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Proposition

Let $\mathbf{A}$ be a bounded involutive lattice. Then the operation $\cdot$ is residuated iff the operation $\to$ is co-residuated.
Moreover, if $\mathbf{A}$ is a bounded involutive lattice for which the above equivalent conditions hold, then $\mathbf{A}$ is an ortholattice.

Residuated Ortholattices

Definition

A *(Sasaki)* *residuated ortholattice* (or *ROL*) is an expansion of an ortholattice $(A, \wedge, \vee, \neg, 0, 1)$ by a binary operation $\backslash$ satisfying

$$x \cdot y \leq z \iff y \leq x\backslash z \tag{R}$$

where $\cdot$ is the Sasaki product: $x \cdot y = x \wedge (\neg x \vee y)$.

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Properties

- Residuated ortholattices form a variety ROL.
- OML is a subvariety of ROL (taking \ to be the Sasaki hook $\rightarrow$).

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Fine spectrum for ROL and OML

n	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
OMLs	1	0	1	0	1	0	2	0	2	0	3
ROLs	1	0	1	0	2	0	4	0	7	0	15

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Theorem (Fussner & S. 2021)

Residuated ortholattices are the equivalent algebraic semantics of their 1-assertional logic.

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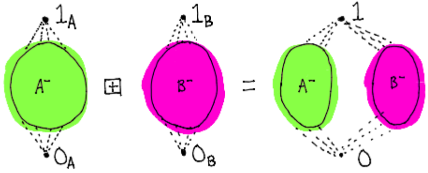
Theorem (Fussner & S. 2022+)

ROL *is closed under* horizontal sums.



$[X^- := X \setminus \{0_X, 1_X\}]$

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Theorem (Fussner & S. 2021)

OML *enjoys a Kolmogorov-style translation into* ROL.

Corollary (Fussner & S. 2021)

OML *has a decidable equational theory if any variety of residuated ortholattices that contains it has a decidable equational theory.*

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Properties

- Both $\cdot$ and \ or order-preserving in their right-coordinates.
- $\cdot$ distributes over arbitrary joins, when they exist, from the left.
- $\cdot$ is idempotent, alternative, and flexible.
- \ distributes over arbitrary meets, when they exist, from the left.

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Properties

- Generally, $\cdot$ is not order-preserving in its left-coordinate.
- Generally, $\backslash$ is not order-reversing in its left-coordinate.
- Generally, $\cdot$ is neither commutative nor associative.

Why focus on associativity relations for Sasaki product?

- The lack of associativity is a huge obstacle in the proof theory of some quantum logics. E.g., the calculus for orthmodular groupoids [Fazio, Ledda, Paoli, S. 2021].
- In our existing work, some associativity relations were crucial for proving the Kolmogorov translation.
- As we will see, assuming full associativity yields far more familiar structures, and are more tractable to deal with.
- Associative ROLs are the simplest starting case and are a promising source for generating the theory.

Theorem

Let $\mathbf{A}$ *be an ROL and* $a, b, c \in A$. *Then any of the following conditions ensure that* (a, b, c) *associates:*

- $a \leq b$
- $a \leq c$
- $b \leq c$

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Corollary (Fussner & S. 2022+)

In ROL (and hence OML), finite products consisting only of variables x and y, where x is the left-most *variable, are equal to $x \cdot y$.*

E.g.,

$$\mathsf{ROL} \vDash x \cdot y \approx (x \cdot ((y \cdot x) \cdot y)) \cdot (y \cdot x)$$

A new negation and a skeleton

Given a residuated ortholattice $\mathbf{A} = (A, \wedge, \vee, \neg, \backslash, 0, 1)$, we define:

$$\sim x := x \backslash 0 \qquad\qquad \overline{x} := \sim\sim x$$

$$\overline{A} := \{\overline{a} : a \in A\}$$

Theorem (Fussner & S. 2021)

Let $\mathbf{A} = (A, \wedge, \vee, \neg, \backslash, 0, 1)$ *be a residuated ortholattice.*

(1) $(\overline{A}, \wedge, \vee, \sim, 0, 1)$ *is an OML, denoted* $\mathsf{OML}(\mathbf{A})$.

(2) $\overline{x} \backslash \overline{y} = \sim(x \cdot \neg y)$ *for all* $x, y \in A$.

(3) *The map* $x \mapsto \overline{x}$ *is an ortholattice homomorphism of* $\mathbf{A}$ *onto* $\mathsf{OML}(\mathbf{A})$.

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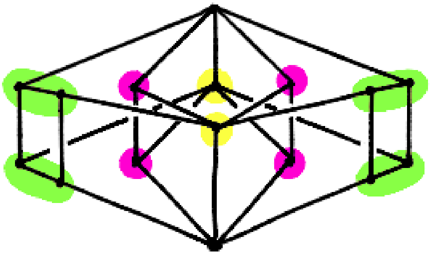
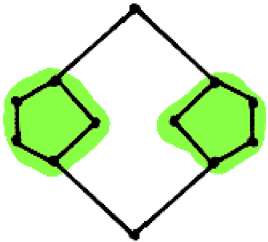
Corollary

Let $\mathbf{A}$ be a residuated ortholattice. Then the following are equivalent:

(1) $\mathbf{A}$ *is an OML.* (2) $\mathbf{A} \vDash \sim x \approx \neg x$. (3) $\mathbf{A} \vDash x \approx \sim\sim x$.

Examples



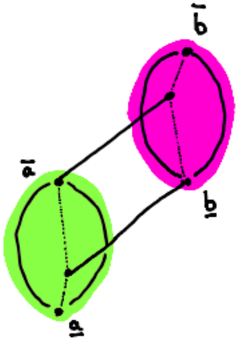


The closure *bubbles*

Proposition

Let $\mathbf{A}$ be an ROL. Then for all $a, b, c \in A$ the following hold:

1. $a \leq c \leq \overline{a} \implies a \cdot b \leq c \cdot b$ and $c\backslash b \leq a\backslash b$.
2. $\overline{a} \leq \overline{b} \iff \underline{a} \leq \underline{b}$, where $\underline{x} := \neg{\sim}x$.



Proposition

Residuated ortholattices satisfy the following quasi-identities:

1. $\overline{a} \approx \overline{b} \implies a \cdot x \approx a \wedge (\neg b \vee x)$
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Proposition

Let $\mathbf{A}$ be an ROL. Then the following hold.

1. $(S_a, \cdot)$ is a left-zero band, i.e. $x \cdot y = x$ for all $x \in S_a$.
2. $(S_a, \wedge, \vee)$ is a sub-lattice of $\mathbf{A}$, with least element $\underline{a}$ and greatest element $\overline{a}$.
3. For all $x \in S_a$ and $x' \in S_{\neg a}$, $x' \backslash x = \overline{a}$
4. For $b \in A$, let ρ_b be the map $x \mapsto x \cdot b$. Then its restriction to S_a is an ℓ-semigroup homomorphism from S_a to S_{ab}.

Theorem (Fussner & S. 2022+)

If any two elements of the set $\{a, b, c\}$ *share the same image under the map* $x \mapsto \overline{x}$*, then the triple* (a, b, c) *associates.*

Let $\mathbf{L}$ be an OML and $a, b, c \in L$.

Definition

We say a commutes with b if $a \cdot b = a \wedge b$.

Commuting elements in OML and associativity

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Fact: An ortholattice is orthomodular if and only if the commuting relation is symmetric.

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Theorem (Foulis-Holland)

If any one of a, b, or c commutes with the other two, then they all commute and the sublattice generated by $\{a, b, c\}$ is a distributive sublattice of $\mathbf{L}$.

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Theorem (Kröger Lemma)

If a commutes with b then $ab \cdot c = a \cdot bc$.

Notions of *commuting elements* in ROL

Definition

Let $\mathbf{A}$ be an involutive lattice and $a, b \in A$. We say:

- a left-commutes with b and (equiv. b right-commutes with a) if $a \cdot b = a \wedge b \quad (\equiv ab \leq b)$
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Definition

Let $\mathbf{A}$ be an involutive lattice. An element $a \in A$ is said to be:

- right-central in $\mathbf{A}$ if it right-commutes with x for all $x \in A$
- left-central in $\mathbf{A}$ if it left-commutes with x for all $x \in A$
- central in $\mathbf{A}$ if it is both right- and left-central in $\mathbf{A}$.

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Fact

In OML, these notions collapse in each definition above.

Let $\mathbf{A}$ be an ROL with $a, b \in A$.

Proposition

The set of all elements for which $\overline{a}$ right-commutes with is closed under the operations $\{\wedge, \vee, \neg, \sim, 0, 1\}$.

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Proposition

- If a is right-central in $\mathbf{A}$ then $a = \overline{a}$.
- If a is left-central in $\mathbf{A}$ then $a = \underline{a}$.
- a is right-central $\iff$ $\neg a$ is left-central in $\mathbf{A}$
- $\overline{a}$ is right-central $\iff$ $\underline{a}$ is left-central in $\mathbf{A}$.
- a is central in $\mathbf{A}$ iff $\underline{a} = \overline{a}$ (i.e., $S_a = \{a\}$) and a is right-central.

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Theorem (Fussner & S. 2022+)

If an ROL $\mathbf{A}$ has a central element c, then $\mathbf{A} \cong [c, 1] \times [\neg c, 1]$.

Boolean skeleton's and commuting elements

Definition

Let $\mathbf{A}$ be an ROL. We say $\mathbf{A}$ has a *Boolean skeleton* if $\mathsf{OML}(\mathbf{A})$ is (term-equivalent to) a Boolean algebra.

Definition

Let $\mathbf{A}$ be an ROL. We say $\mathbf{A}$ has a *Boolean skeleton* if $\mathsf{OML}(\mathbf{A})$ is (term-equivalent to) a Boolean algebra.

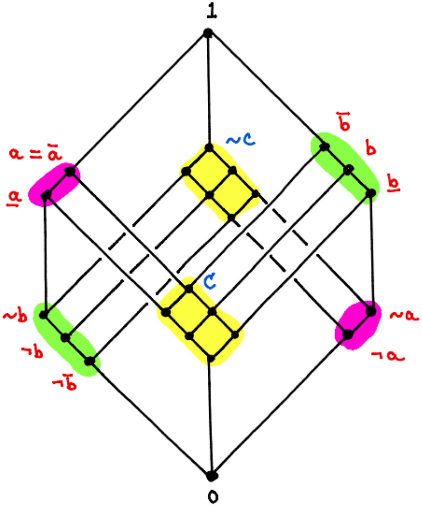
Lemma

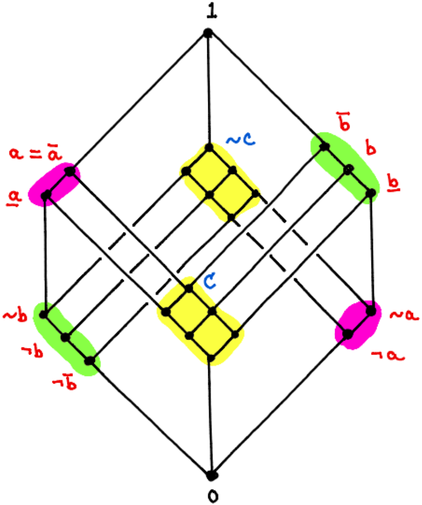
An ROL $\mathbf{A}$ has a Boolean skeleton iff $\overline{a}$ is right-central for all $a \in A$.

Proof.

$$\begin{aligned}
\mathsf{OML}(\mathbf{A}) \text{ is Boolean} \quad &\Longleftrightarrow \quad \mathsf{OML}(\mathbf{A}) \models \quad x \cdot y \approx x \wedge y \\
&\Longleftrightarrow \quad \mathsf{OML}(\mathbf{A}) \models \quad x \cdot y \leq y \\
&\Longleftrightarrow \quad \mathbf{A} \models \quad \overline{x} \cdot \overline{y} \leq \overline{y} \\
&\Longleftrightarrow \quad \mathbf{A} \models \quad x \cdot \overline{y} \leq \overline{y}
\end{aligned}$$

□


1
$a = \bar{a}$
$\underline{a}$
$\sim c$
$\bar{b}$
b
$\underline{b}$
c
$\sim b$
$\neg b$
$\neg \bar{b}$
$\sim a$
$\neg a$
0



Theorem (Fussner & S. 2022+)

Let $\mathbf{A} \in \mathsf{ROL}$ *with right-central elements* a, b. *If* $a \geq {\sim}b$ *then the lattice* $[\underline{a} \vee {\sim}b, a] \times [{\sim}a \vee \underline{b}, b]$ *embeds into* $S_{a \wedge b}$.

The (almost) Kröger Lemma

Lemma

Residuated ortholattices satisfy the following quasi-identity:

$$y \cdot \overline{x} \approx y \wedge \overline{x} \quad \Longrightarrow \quad xy \cdot z \approx x \cdot yz \vee x \cdot (\neg y \vee z) \neg x$$

$$\overline{x} \text{ right-commutes with } y \quad \Longrightarrow \quad xy \cdot z \leq x \cdot yz$$

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\overline{x} \textit{ right-commutes with } y & \Longrightarrow & xy \cdot z \leq x \cdot yz
\end{array}
$$

Theorem (Fussner & S. 2022+)

In an ROL, if $\overline{a}$ right-commutes with both b and c then $ab \cdot c = a \cdot bc$.

Theorem (Fussner & S. 2022+)

Let $\mathbf{A}$ be an ROL. Then the following are equivalent.

- $\mathbf{A}$ *has a Boolean skeleton.*
- $\overline{a}$ *is right-central in* $\mathbf{A}$ *for all* $a \in A$.
- *Sasaki product is associative in* $\mathbf{A}$.
- $\mathbf{A}$ *satisfies* $x(y + z) \approx xy + xz$, *where* $x + y := \neg(\neg x \cdot \neg y)$.

Theorem (Fussner & S. 2022+)

Let $\mathbf{A}$ be an ROL. Then the following are equivalent.

- *$\mathbf{A}$ has a Boolean skeleton.*
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- *$\mathbf{A}$ satisfies $x(y + z) \approx xy + xz$, where $x + y := \neg(\neg x \cdot \neg y)$.*

Theorem (Fussner & S. 2022+)

Let $\mathcal{A}$ be the subvariety of associative ROLs, and let ε be an equation containing only variables and the operation $\cdot$ (Sasaki product). Then

$$\varepsilon \text{ holds in } \mathcal{A} \iff \varepsilon \text{ holds in all left-regular bands}$$

[i.e., idempotent semigroups satisfying $xyx \approx xy$].

Future work

- Further develop the role of (maximal) associative subalgebas of ROLs [e.g., the role of "Boolean blocks", as in OML]
- Generalize the Foulis-Holland theorem to a more general setting.
- Exploit the role this *near-associativity* can be useful for a logical calculus [E.g., the data-type of structures in a sequent calculus].
- Solidify the ties between Substructural Logic and Quantum Logic.

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Thank you!