One-Sorted Program Algebras

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Topology, Algebra and Categories in Logic 2022
Coimbra, Portugal

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- We've also shown that the substructural logic of partial correctness S [KT03] embeds into residuated KAD (called SKAT).

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1. Kleene algebra with tests

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- $(K, +, 0)$ join-semilattice
- $(K, \cdot, 1)$ monoid
- $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$
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and ${}^* : K \to K$ (Kleene star) satisfies

$$1 + x + x^*x^* \leq x^* \tag{1}$$

$$xy \leq y \Rightarrow x^*y \leq y \tag{2}$$

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Examples: Algebras of binary relations, regular languages, matrices over semirings, functions from monoids to complete lattices...

Kleene algebra with tests

$\mathcal{K} = (K, B, \cdot, +, {}^*, -, 1, 0)$

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- $B \subseteq K$
- $(B, \cdot, +, -, 1, 0)$ Boolean algebra

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Propositional while programs

- if b then p else q: $(bp) + (\bar{b}q)$, while b do p: $(bp)^*\bar{b}$
- $\{b\}p\{c\}$: $bp\bar{c} = 0$

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Theorem. *The eq. theory of KAT is PSPACE-complete [CKS96], and the Horn theory with assumptions $r = 0$ reduces to the eq. theory [KS97].*

2. Kleene algebra with (co)domain

Kleene algebra with (co)domain

The idea: Expand $\mathcal{K} = (K, \cdot, +, ^*, 1, 0)$ with unary t and a such that

$$\mathsf{t}(\mathcal{K}) = (\mathsf{t}(K), \cdot, +, \mathsf{a}, 1, 0)$$

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Inspiration:

$$\mathsf{d}(R) = \{(s, s) \mid \exists t\, R(s, t)\} \qquad \mathsf{c}(R) = \{(t, t) \mid \exists s\, R(s, t)\}$$

Kleene algebra with (co)domain

KAD: $\mathcal{K} = (K, \cdot, +, {}^*, 1, 0, \mathsf{d}, \mathsf{a})$ where $(K, \cdot, +, {}^*, 1, 0)$ is KA and

$$x \leq \mathsf{d}(x)x \tag{4}$$

$$\mathsf{d}(xy) = \mathsf{d}(x\mathsf{d}(y)) \tag{5}$$

$$\mathsf{d}(x) \leq 1 \tag{6}$$

$$\mathsf{d}(0) = 0 \tag{7}$$

$$\mathsf{d}(x + y) = \mathsf{d}(x) + \mathsf{d}(y) \tag{8}$$

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KAC: A "symmetric variant" with c instead of d.

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Theorem. *The quasi-equational theory of KAT embeds into the quasi-equational theory of KAD (and KAC).*

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Proof. This follows from:

1. $\mathsf{d}(\mathcal{A})$ is a subalgebra of $\mathcal{A}$
2. $(\mathsf{d}(A), \cdot, +, 1, 0)$ is a bounded distributive lattice (since $\mathsf{d}(x) \leq 1$ and $\mathsf{d}(x)\mathsf{d}(x) = \mathsf{d}(x)$)
3. $\mathsf{a}(\mathsf{d}(x))$ is a complement of $\mathsf{d}(x)$. □

Kleene algebra with (co)domain

Let $\Gamma \cup \varphi$ be a set of equations over $\mathcal{L}_{KAT}$.

Theorem 1. *There is a function $Tr : \mathcal{L}_{KAT} \to \mathcal{L}_{KAD}$ such that KAT $\models \Gamma \Rightarrow \varphi$ iff KAD $\models Tr(\Gamma) \Rightarrow Tr(\varphi)$. (Similarly for KAC.)*

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Proof. Let $Tr(\mathsf{p}_n) = \mathsf{x}_{2n}$, $Tr(\mathsf{b}_n) = \mathsf{d}(\mathsf{x}_{2n+1})$ and $Tr(\bar{b}) = \mathsf{a}(Tr(b))$, while Tr commutes with the KA operators. We discuss the case $\Gamma = \emptyset$.

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1. If KAT $\not\models p \approx q$, then there is a full relational $\mathcal{R} \not\models p \approx q$ [KS97], i.e. $[p] \neq [q]$ for some valuation $[\,]$. By Lemma 1, $\mathcal{R}$ is a KAD. Define $[\![\,]\!]$ as the unique KAD-valuation such that $[\![\mathsf{x}_{2n}]\!] = [\mathsf{p}_n]$ and $[\![\mathsf{x}_{2n+1}]\!] = [\mathsf{b}_n]$.

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Claim. For all $p \in \mathcal{L}_{KAT}$, $[p] = [\![Tr(p)]\!]$.

(Note that $[\mathsf{b}_n] \in B$ and so $[\mathsf{b}_n] = \mathsf{d}\,[\mathsf{b}_n] = \mathsf{d}[\![\mathsf{x}_{2n+1}]\!] = [\![Tr(\mathsf{b}_n)]\!]$.

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Hence, KAT $\not\models p \approx q$.

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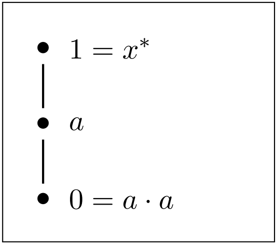

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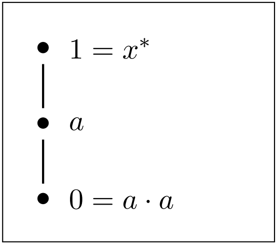

If there is a d, then $\mathsf{d}(a) \in \{a, 1\}$.

- If $\mathsf{d}(a) = a$, then $\mathsf{d}(a)a = 0$ and so $a \not\leq \mathsf{d}(a)a$ (¬4).
- If $\mathsf{d}(a) = 1$, then $\mathsf{d}(a\mathsf{d}(a)) = 1 \neq 0 = \mathsf{d}(aa)$ (¬5).

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Proof. ([DS11]). It can be shown that $\mathsf{d}(x) = x$ for every x such that $\exists y(yx = 0 \ \& \ x + y = 1)$, using

1. $x \leq x\mathsf{d}(x)$
2. $\mathsf{d}(x) \leq 1$
3. $\mathsf{d}(y\,\mathsf{d}(x)) \leq \mathsf{d}(yx)$ $\square$

3. One-sorted KAT

Generalizing KAD

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Proof. This follows from:

1. $\mathsf{d}(\mathcal{A})$ is a subalgebra of $\mathcal{A}$
2. $(\mathsf{d}(A), \cdot, +, 1, 0)$ is a bounded distributive lattice (since $\mathsf{d}(x) \leq 1$ and $\mathsf{d}(x)\mathsf{d}(x) = \mathsf{d}(x)$)
3. $\mathsf{a}(\mathsf{d}(x))$ is a complement of $\mathsf{d}(x)$.

Generalizing KAD

Recall **Lemma 2**: *If $\mathcal{A}$ is KAD, then* $\mathsf{d}(\mathcal{A})$ *is BA.*

Proof. This follows from:

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Question: Is this possible without $\mathsf{d}(y\,\mathsf{d}(x)) \leq \mathsf{d}(yx)$ (or $x \leq \mathsf{d}(x)$)?

OneKAT

$\mathcal{K} = (K, \cdot, +, {}^*, 1, 0, \mathsf{t}, \mathsf{a})$ where $(K, \cdot, +, {}^*, 1, 0)$ is KA and

$$\mathsf{t}(0) = 0 \tag{11}$$

$$\mathsf{t}(1) = 1 \tag{12}$$

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Proposition 1. *Every KA expands into a OneKAT.*

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Proof.

$$\mathsf{t}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise.} \end{cases} \qquad \mathsf{a}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ x & \text{otherwise.} \end{cases} \qquad \square$$

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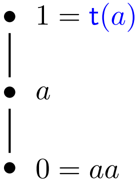
Proposition 2. *The test algebra* $\mathsf{t}(\mathcal{A}) = (\mathsf{t}(A), \cdot, +, \mathsf{a}, 1, 0)$ *is not necessarily the largest Boolean subalgebra of the negative cone of the KA underlying* $\mathcal{A}$*.*

OneKAT and KAD



$\mathtt{t}(a\,\mathtt{t}(a)) \not\leq \mathtt{t}(a\,a)$

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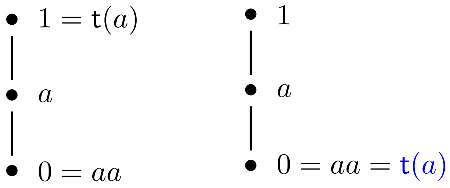

$\mathsf{t}(a\,\mathsf{t}(a)) \not\leq \mathsf{t}(a\,a)$ $\qquad$ $a \not\leq a\,\mathsf{t}(a)$

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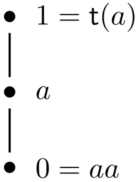

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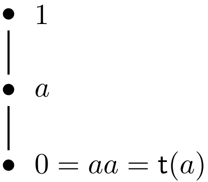

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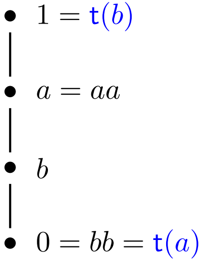

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OneKAT and KAD

Lemma 3. *Every KAD (and KAC) is a OneKAT.*

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Proof. By definition of OneKAT:

1. $\mathtt{t}(\mathcal{A}) = (\mathtt{t}(A), \cdot, +, \mathsf{a}, 1, 0)$ is a subalgebra of $\mathcal{A}$;
2. $(\mathtt{t}(A), \cdot, +, 1, 0)$ is a bounded distributive lattice;
3. $\mathsf{a}(\mathtt{t}(x))$ is a complement of $\mathtt{t}(x)$. □

A related generalization of KAD

A few days ago we've been notified about [AGS16] where a related generalization is briefly mentioned:

$\mathcal{A} = (A, \cdot, +, {}^*, \mathsf{n}, 1, 0)$, where $\mathsf{t}(x) := \mathsf{n}(\mathsf{n}(x))$ and

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This generalization has all the good properties of OneKAT.

4. KAT embeds into OneKAT

KAT and OneKAT

Theorem 2. *There is a function* $Tr : \mathcal{L}_{KAT} \to \mathcal{L}_{OneKAT}$ *such that* KAT $\models \Gamma \Rightarrow \varphi$ *iff* OneKAT $\models Tr(\Gamma) \Rightarrow Tr(\varphi)$*. (Similarly for KAC.)*

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Proof. Tr is defined as before. We reason for $\Gamma = \emptyset$. By Theorem 1, if KAT $\not\models p \approx q$, then KAD $\not\models Tr(p) \approx Tr(q)$ and so by Lemma 3, OneKAT $\not\models Tr(p) \approx Tr(q)$.

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If OneKAT $\not\models Tr(p) \approx Tr(q)$, then $[\![p]\!]_{\mathcal{A}} \neq [\![q]\!]_{\mathcal{A}}$ where $\mathcal{K} = (A, \mathsf{t}(A), \cdot, +, {}^*, \mathsf{a}, 1, 0)$ is a KAT by Lemma 4.

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If OneKAT $\not\models Tr(p) \approx Tr(q)$, then $[\![p]\!]_{\mathcal{A}} \neq [\![q]\!]_{\mathcal{A}}$ where $\mathcal{K} = (A, \mathsf{t}(A), \cdot, +, {}^*, \mathsf{a}, 1, 0)$ is a KAT by Lemma 4.

We define $[\,]_{\mathcal{K}}$ as before and prove that $[\![Tr(p)]\!]_{\mathcal{A}} = [p]_{\mathcal{K}}$ for all p as before. It follows that KAT $\not\models p \approx q$. $\square$

5. SKAT and an embedding of S

The logic S [KT03]

Let $\mathsf{B} = \{\mathsf{b}_i \mid i \in \omega\}$ be the set of test variables and let $\mathsf{P} = \{\mathsf{p}_i \mid i \in \omega\}$ be the set of program variables. Let $\mathsf{E} = \mathsf{B} \cup \mathsf{P}$

- tests $\quad b, c := \mathsf{b}_i \mid 0 \mid b \Rightarrow c$
- programs $\quad p, q := \mathsf{p}_i \mid b \mid p \oplus q \mid p \otimes q \mid p^+$
- formulas $\quad f, g := b \mid p \Rightarrow f$
- environments $\quad \Gamma, \Delta := \epsilon \mid \Gamma, p \mid \Gamma, f$
- sequents $\quad \Gamma \vdash f$

Let Ex_{S} be the union of the sets of formulas, programs and environments.

The logic S [KT03]

A **Kozen–Tiuryn model** is a pair $M = (W, V)$ where V : E $\rightarrow 2^{W \times W}$ such that $V(\mathsf{b}) \subseteq \mathrm{id}_W$.

For each M, we define the **M-interpretation** function $[\,]_M : Ex_{\mathsf{S}} \rightarrow 2^{W \times W}$ as follows:

- $[\mathsf{b}]_M = V(\mathsf{b})$, $[\mathsf{p}]_M = V(\mathsf{p})$ and $[0]_M = \emptyset$
- $[b \Rightarrow c]_M = \{(s,s) \mid (s,s) \notin [b]_M$ or $(s,s) \in [c]_M\}$
- $[p \oplus q]_M = [p]_M \cup [q]_M$ and $[p \otimes q]_M = [p]_M \circ [q]_M$
- $[p^+]_M = [p]_M^+$
- $[p \Rightarrow f]_M = \{(s,s) \mid \forall t.(s,t) \in [p]_M \implies (t,t) \in [f]_M\}$
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A sequent $\Gamma \vdash f$ is valid in M iff, for all $s, t \in W$, if $(s, t) \in [\Gamma]_M$, then $(t, t) \in [f]_M$.

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A SKAT is $(K, \cdot, +, \to, \hookrightarrow, {}^*, \mathsf{t}, \mathsf{e}, 1, 0)$ where $(K, \cdot, +, \to, \hookrightarrow, {}^*, 1, 0)$ is a residuated Kleene algebra, and t and e satisfy the following:

$$\mathsf{t}(\mathsf{t}(x)\mathsf{t}(y)) = \mathsf{t}(x)\,\mathsf{t}(y) \tag{14}$$
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Theorem 3. *There is a function $Tr : Ex_{\mathsf{S}} \to \mathcal{L}_{SKAT}$ such that $\Gamma \vdash f$ is valid in all KT models iff $\mathtt{t}(Tr(\Gamma)) \leq Tr(f)$ is valid in all *-continuous SKAT.*

Conclusion

OneKAT is a generalization of KAD (and KAC) that keeps (some of) their good properties while it avoids the bad properties, namely:

- KAT embeds into OneKAT
- Every KA expands into a OneKAT
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Future work:

- Free OneKAT? (Generalising [McL20])
- PSPACE-complete?

Thank you!

References I

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