Epimorphisms in varieties of De Morgan monoids[1]

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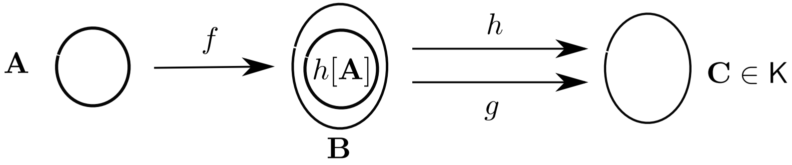
Epimorphisms

Let K be a variety of algebras and $\mathbf{A}, \mathbf{B} \in$ K. A homomorphism $f : \mathbf{A} \to \mathbf{B}$ is an *epimorphism* if, whenever $\mathbf{C} \in$ K and $g, h : \mathbf{B} \to \mathbf{C}$ are homomorphisms,

$$\text{if } g \circ f = h \circ f, \text{ then } g = h.$$

All surjective homomorphisms are epimorphisms, but, the converse need not be true.



Epic subalgebras

We say K has the *epimorphism surjectivity (ES) property* if all its epimorphisms are surjective.

The ES property is not in general inherited by subvarieties.

A subalgebra $\mathbf{A} \leqslant \mathbf{B} \in \mathsf{K}$ is *epic* if the inclusion map $\mathbf{A} \hookrightarrow \mathbf{B}$ is an epimorphism, i.e., homomorphisms from $\mathbf{B}$ to members of K are determined by their restrictions to A.

Let K be a variety of algebras. The following are equivalent:

- K lacks the ES property.
- There is a member of K with a *proper* epic subalgebra.

Thm (Campercholi 2018) Let K be an arithmetical variety whose finitely subdirectly irriducible (FSI) members form a universal class. Then K has the ES property iff its *FSI members* lack proper epic subalgebras.

Beth property

Let K be a variety of algebras that algebraizes a logic $\vdash$. The following are equivalent:

- K has surjective epimorphisms.
- $\vdash$ satisfies the *infinite Beth property*, i.e., all *implicit* definitions of propositional functions in $\vdash$ can be made *explicit*.

De Morgan monoids

A *De Morgan monoid* $\mathbf{A} = \langle A; \wedge, \vee, \cdot, \neg, e \rangle$ comprises

- a distributive lattice $\langle A; \wedge, \vee \rangle$,
- a commutative monoid $\langle A; \cdot, e \rangle$ that is *square-increasing* $(x \leqslant x^2 := x \cdot x)$
- an involution $\neg$ satisfying $\neg\neg x = x$ and

$$x \cdot y \leqslant z \text{ iff } x \cdot \neg z \leqslant \neg y.$$

One can define a *residual* $x \to y :\ = \neg(x \cdot \neg y)$ satisfying the *law of residuation*

$$x \cdot y \leqslant z \text{ iff } x \leqslant y \to z,$$

and the constant $f := \neg e$.

The class DMM of all De Morgan monoids form a variety.

Dunn monoids are the $\neg$-less $(\wedge, \vee, \cdot, \to, e)$ subreducts of De Morgan monoids.

Negative generation

An element a of a De Morgan monoid $\mathbf{A}$ is *negative* if $a \in A^- := \{b \in A \colon b \leqslant e\}$. And $\mathbf{A}$ is said to be *negatively generated* if $\mathbf{A} = \mathrm{Sg}(A^-)$.

- In any Dunn/De Morgan monoid if $x, y \leqslant e$ then $x \cdot y = x \wedge y$.
- A *Brouwerian algebra* is a *integral* ($x \leqslant e$) Dunn monoid in which $\cdot$ coinsides with $\wedge$.
- The *negative cone* A^- can be turned into a Brouwerian algebra $\mathbf{A}^- = \langle A^-; \wedge, \vee, \to^-, e\rangle$, by restricting $\wedge, \vee$ to A^- and defining

$$a \to^- b = (a \to b) \wedge e, \text{ for } a, b \in A^-.$$

Depth

Let $\mathbf{A}$ be De Morgan monoid.

- A set $F \subseteq A$ is a *deductive filter* of $\mathbf{A}$ if it is a lattice filter which contains e.
- A filter F is *prime* if $F = A$ or $A \setminus F$ is a lattice ideal.
- Let $\mathrm{Pr}(\mathbf{A})$ be the set of prime deductive filters of $\mathbf{A}$.
- We say that $\mathbf{A}$ has *infinite depth* if its poset of prime deductive filters contains an infinite descending chain; otherwise it has *finite depth*.

Theorem 1

Let K be a variety of De Morgan monoids such that for every FSI member **A** of K

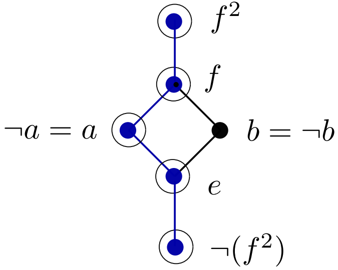
1) $\mathbf{A} = \mathrm{Sg}(A^-)$, and
2) **A** has finite depth.

Then K has surjective epimorphisms.

- The proof of this theorem does not require distributivity.
- It holds in the presence of bounds and for $\neg$-less subreducts (so that it generalizes a result in Bezhanishvili, M. and R. (2017) which concerned Brouwerian (and Heyting) algebras).

Condition (1) can't be dropped

Urquhart (1999) showed that the *crystal lattice* has a proper epic subalgebra in DMM. Notice that it is not negatively generated.

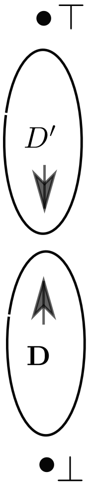


Reflection

Let $\mathbf{D}$ be a Dunn monoid.

- There is a unique way of turning the structure into a De Morgan monoid $R(\mathbf{D}) = \langle D \cup D' \cup \{\bot, \top\}; \wedge, \vee, \cdot, \neg, e\rangle$ of which $\mathbf{D}$ is a subreduct and where $\neg$ extends $'$.
- If K is a class of Dunn monoids we define $\mathbb{R}(\mathsf{K}) := \mathbb{V}(\{R(\mathbf{D}) : \mathbf{D} \in \mathsf{K}\})$.

$R(\mathbf{D})$:



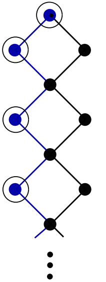
Properties of reflections

Let $\mathbf{D}$ be a Dunn monoid and K a variety of Dunn monoids.

- $\mathbf{D}$ is negatively generated iff $R(\mathbf{D})$ is.
- K is locally finite iff $\mathbb{R}(\mathsf{K})$ is.
- K has surjective epimorphisms iff $\mathbb{R}(\mathsf{K})$ has. In particular, $\mathbf{D}$ has a proper epic subalgebra in K iff $R(\mathbf{D})$ has a proper epic subalgebra in $\mathbb{R}(\mathsf{K})$.
- The map $\mathsf{K} \mapsto \mathbb{R}(\mathsf{K})$ from the subvariety lattice of Dunn monoids into the subvariety lattice of De Morgan monoids is *injective*.

Condition (2) can't be dropped

Bezhanishvili, M., and R. exhibits a Brouwerian algebra $\mathbf{A}_1$ that generates a variety without the ES property.
$R(\mathbf{A}_1)$ has infinite depth and has a proper $\mathbb{R}(\mathbf{A}_1)$-epic subalgeba.

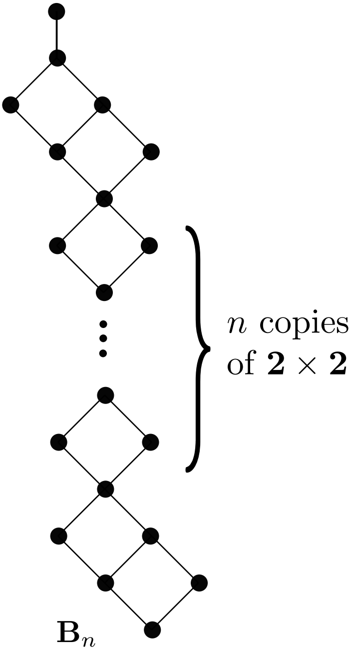


$\mathbf{A}_1$

For every $n \in \omega$, consider the depicted Brouwerian algebra $\mathbf{B}_n$. Let $F := \{\mathbf{B}_n : n \in \omega\}$. Adapting Bezhanishvili[2] and de Jongh (2008): for every different pair $T, S \subseteq F$, we get $\mathbb{V}(T) \neq \mathbb{V}(S)$.
For every $T \subseteq F$, we show that $\mathbb{V}(T, \mathbf{A}_1)$ is locally finite and fails to have the ES property, and the map

$$\mathbb{V}(T) \mapsto \mathbb{V}(T, \mathbf{A}_1)$$

is injective.



Using reflections we get:

Thm

There is $2^{\aleph_0}$ (locally finite) varieties of De Morgan monoids (of infinite depth) that don't have the ES property.

Semilinearity

A De Morgan monoid will be called *semilinear* if it is a subdirect product of totally ordered algebras. These form a variety axiomatized by $e \leqslant (x \to y) \vee (y \to x)$.

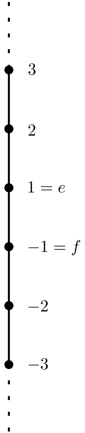
Recall that *idempotent* ($x = x \cdot x$) De Morgan monoids are called *Sugihara monoids*. They are negatively generated generated and form a locally finite variety.

The variety of *Sugihara monoids* is generated by the algebra $\mathbf{Z}^*$, which is the natural chain of nonzero integers (with $-$ as $\neg$),

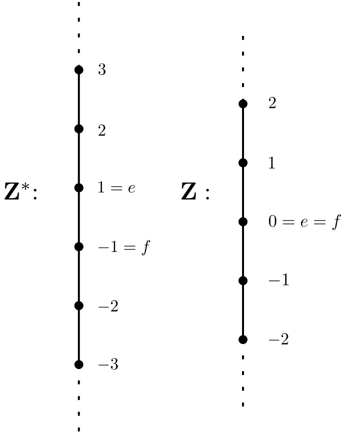
$$x \cdot y = \begin{cases} \text{whichever of } x, y \text{ has} \\ \quad \text{a greater absolute value} \\ \text{or } x \wedge y \text{ if } |x| = |y|. \end{cases}$$

Here, $e = 1$, so $f = -1$.

$\mathbf{Z}^*$



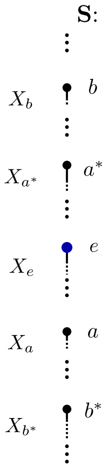
$\mathbf{Z}^*$ has a homomorphic image $\mathbf{Z}$ in which just 1 and -1 are identified.
Up to isomorphism, $\mathbf{Z}$ could be defined like $\mathbf{Z}^*$ on the set of *all* integers.
Then $\mathbf{Z} \models f = e\,(= 0)$.
Algebras with $f = e$ are called *odd*.
The variety of odd Sugihara monoids (OSM) is generated by $\mathbf{Z}$.



Let **S** be a totally ordered odd Sugihara monoid and let $\mathcal{X} = \{\mathbb{X}_c : c \in S\}$ such that each $\mathbb{X}_c$ is a chain with greatest element c.
Let $\mathbf{S} \otimes \mathcal{X}$ denote the algebra on $\bigcup\{X_c : c \in S\}$ with the lexicographic total order, where for $a, b \in S$ and $x \in X_a, y \in X_b$,

$$x \cdot y = \begin{cases} x \wedge y & \text{if } a = b \leqslant e \\ x \vee y & \text{if } e < a = b \\ x & \text{if } a \neq b \text{ and } a \cdot^{\mathbf{S}} b = a \\ y & \text{if } a \neq b \text{ and } a \cdot^{\mathbf{S}} b = b \end{cases}$$

$$x \to y = \begin{cases} (a \to e) \vee y & \text{if } x \leqslant y \\ (a \to e) \wedge y & \text{if } y < x \end{cases}.$$



Thm (Gil-Férez, Jipsen and Metcalfe, 2020) $\mathbf{S} \oplus \mathcal{X}$ is a totally ordered idempotent Dunn monoid. Moreover, every totally ordered idempotent Dunn monoid has this form.

Generalized Sugihara monoids are negatively generated semilinear idempotent Dunn monoid, and they form a locally finite variety GSM (Raftery, 2007).

The totally ordered members are $\mathbf{S} \oplus \mathcal{X}$ as above, but $X_c = \{c\}$ for every $e \leqslant c \in S$.

Thm If a totally ordered Dunn monoid is generated by idempotent elements, then it is idempotent.

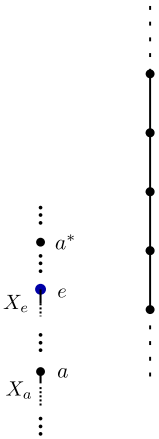
Therefore, every variety of negatively generated semilinear Dunn monoids is a subvariety of GSM.

Thm (M., R. and W., 2019)

A totally ordered De Morgan monoid **A** is negatively generated iff

1) **A** is a Sugihara monoid, or
2) $A = (\bot] \cup R(\mathbf{D}) \cup [\top)$ where $(\bot]$ and $[\top)$ are chains of idempotents, and **D** is a totally ordered member of GSM.

This can be used to show that the class of negatively generated semilinear De Morgan monoids is a locally finite variety.



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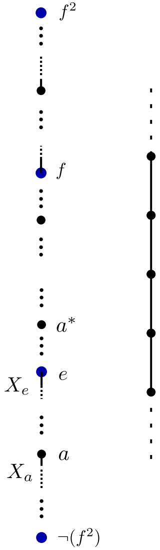

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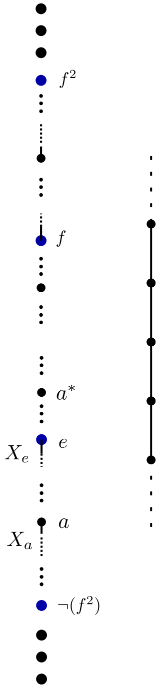


Theorem 2

Every variety of negatively generated semilinear De Morgan monoids has surjective epimorphisms.

The same is true for generalized Sugihara monoids.

This result therefore generalizes the following results, see (Bezhanishvili, M. and R., 2017):

- Every variety of semilinear Brouwerian algebras (*relative Stone algebras*) has surjective epimorphisms.
- Every variety of Sugihara monoids has surjective epimorphisms,
- the same applies to the involution-less subreducts of Sugihara monoids.
- Every variety of generalized Sugihara monoids has a weak version of the ES property (Galatos and Raftery, 2015).

Negative generation can't be dropped

For every $n \geq 1$, a De Morgan monoid $\mathbf{A}_n$ can be defined on

$$0 < 1 < 2 < 2^2 < \cdots < 2^n < 2^{n+1},$$

where $\cdot$ is integer multiplication truncated at 2^{n+1}, and $\neg 2^m = 2^{n-m}$ for $0 \leq m \leq n$.

$\{0, 1, 2^n, 2^{n+1}\}$ is a proper epic subalgebra of $\mathbf{A}_n$ in *every* variety of semilinear De Morgan monoids containing $\mathbf{A}_n$.

For two different sets P, Q of prime numbers,

$$\mathbb{V}(\{\mathbf{A}_p : p \in P\}) \neq \mathbb{V}(\{\mathbf{A}_q : q \in Q\}).$$

Thm

The ES property fails in $2^{\aleph_0}$ varieties of semilinear De Morgan monoids.

Theorems 1 and 2 do not encompass all varieties of De Morgan monoids with the ES property. E.g., take the reflection of the following (not neg. gen.) varieties.

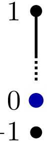
Thm

Epimorphisms are surjective in the variety of semilinear idempotent Dunn monoids.

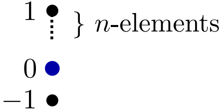
Similarly for the next examples.

Let S be the set of simple semilinear idempotent Dunn monoids (see their form below).

simple sli Dunn monoid:



$\mathbf{C}_n$:



Simples

The subvariety lattice of $\mathbb{V}(S)$ has the following form

$$\mathbb{V}(\mathbf{S}_3) = \mathbb{V}(\mathbf{C}_1) \subsetneq \mathbb{V}(\mathbf{C}_2) \subsetneq \mathbb{V}(\mathbf{C}_3) \subsetneq \ldots \mathbb{V}(S).$$

Thm

The subvarieties of $\mathbb{V}(S)$ with the ES property are exactly $\mathbb{V}(S)$ and $\mathbb{V}(\mathbf{S}_3)$.

Proof

Claim: the automorphism group of $\mathbf{C}_n$ is trivial.
We show that the integer subalgebra of $\mathbf{C}_n$ is epic in $\mathbb{V}(\mathbf{C}_n)$. Let $\mathbf{C}$ be a subdirectly irreducible member of $\mathbb{V}(\mathbf{C}_n)$, and $g, h : \mathbf{C}_n \to \mathbf{C}$ agree on the integers. W.l.o.g. g and h are non-trivial. Since $\mathbf{C}_n$ is simple, g, h are embeddings. So, by Jonsson's theorem, $\mathbf{C} = \mathbf{C}_n$. So, g, h are automorphisms, which are equal (to the identity map) by the claim.

thank you