Unitless Frobenius quantales[1]

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Plan

On the definition of Frobenius quantales

Girard quantales from tight sup-preserving maps

Nuclei and phase quantales

Unital Frobenius quantales from pregroups

Unitless Girard quantales from $\mathbb{C}^*$-algebras.

No adding units

A little context: linear orders on Girard quantales

- Linear orders valued in $\mathbf{2} = [C_2, C_2]$.
- Linear orders valued in $[C_n, C_n]$.
- Linear orders valued in $[\,[0,1]\,,\,[0,1]\,]$.

- Linear orders valued in $[L, L]$?
- When $[L, L]$ is a Girard/Frobenius quantale?
- Units are an obstacle to define linear orders valued on a Girard quantale Q.
- Morphisms of Girard quantales that do not preserve units.

Quantales, definition

Definition. A *quantale* is a pair $(Q, *)$ where Q is a complete lattice and $*$ is a semigroup operation that distributes over arbitrary suprema, in each variable:

$$(\bigvee_{i \in I} x_i) * (\bigvee_{j \in J} y_j) = \bigvee_{i \in I, j \in J} x_i * y_j \,,$$

for each pair of families $\{ x_i \mid i \in I \}$ and $\{ y_j \mid j \in J \}$. If the semigroup operation $*$ has a unit, then we say that the quantale is *unital*.

Implications/residuals/adjoints:

$$x \backslash z := \bigvee \{ y \mid x * y \leq z \} \,, \qquad y / z := \bigvee \{ x \mid x * y \leq z \} \,,$$

so

$$x * y \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad x \leq z / x \,.$$

Frobenius quantales, via dualizing elements

Definition. Let $(A, *)$ be a quantale. An element $0 \in Q$ is *dualizing* if, for every x in Q, we have

$$0/(x\backslash 0) = (0/x)\backslash 0 = x\,.$$

The element 0 is *cyclic* if for every x in Q we have

$$x\backslash 0 = 0/x\,.$$

A *Frobenius quantale* is a tuple $(Q, *, 0)$ where $(Q, *)$ is a quantale and $0 \in Q$ is dualizing. If moreover 0 is cyclic, then $(Q, *, 0)$ is a *Girard quantale*.

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First statement in the theory of Frobenius quantales:

Theorem. Any Frobenius quantale is unital.

Frobenius quantales, via negations

Definition. A *Frobenius quantale* is a quantale $(Q, *)$ equipped with a Serre duality,[2] that is, a pair of inverse antitone maps ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q$ satisfying

$$x \backslash {}^{\perp}y = x^{\perp}/y\,, \qquad \text{for every } x, y \in Q. \tag{1}$$

A *Girard quantale* is a Frobenius quantale with ${}^{\perp}(-) = (-)^{\perp}$.

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Remark. Equation (1), known in [Galatos et al., 2007] as the *law of contraposition*, amounts to the *associative law*:

$$x * y \leq {}^{\perp} z \quad \text{iff} \quad x^{\perp} \geq y * z \quad \text{iff} \quad x \leq {}^{\perp}(y * z) \,,$$

and to the *shift relations*:

$$x * y \leq z \quad \text{iff} \quad {}^{\perp} z * x \leq {}^{\perp} y \quad \text{iff} \quad y * z^{\perp} \leq x^{\perp} \,.$$

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With this definition :
Lemma. A Frobenius quantale is unital if and only if it has a dualizing element.

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Are unitless Frobenius quantales useful/interesting?

- A trivial example of unitless Girard quantale:
  the Chu construction (i.e. Twist product) of a unitless quantale.
- Other examples?
- Do they have a nice theory?

Plan

Girard quantales of sup-preserving maps

Let $[L, L]$ be the complete lattice of sup-preserving endomaps of L.

Theorem [Kruml and Paseka, 2008, Egger and Kruml, 2010, Santocanale, 2020b, Santocanale, 2020a].
The quantale $([L, L], \circ)$ has a dualizing element if and only if L is completely distributive.

Beyond complete distributivity: tight maps

Let L be a complete lattice. The Raney's transforms are

$$f^{\vee}(x) := \bigvee_{x \not\leq t} f(t)\,, \qquad\qquad f^{\wedge}(x) := \bigwedge_{t \not\leq x} f(t)\,,$$

where $f : L \longrightarrow L$ is an arbitrary map. Remark: $(-)^{\vee} \dashv (-)^{\wedge}$.

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Definition. Cf. [Raney, 1960].
A sup-preserving map $f : L \longrightarrow L$ is *tight* if $f^{\wedge\vee} = f$.
We let $[L, L]_{\mathtt{t}}$ be the set of tight endomaps of L.

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Remark. Cf. [Wille, 1985, Grätzer and Wehrung, 1999]. We have

$$[L, L]_{\mathtt{t}} = (L^{op} \otimes_{\mathtt{Wille}} L)^{op} =_f (L^{op} \Box L)^{op}$$

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Theorem. For *any* complete lattice L, the quantale $([L, L]_{\mathtt{t}}, \circ)$ is Girard with negation given by

$$f^{\perp} := \ell(f^{\wedge}) = \rho(f)^{\vee}\,.$$

Girard quantales of sup-preserving maps

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Theorem (Egger, Kruml, Paseka, Santocanale).
The quantale $([L, L], \circ)$ has a dualizing element if and only if L is completely distributive.

Theorem. The quantale $([L, L]_{\mathtt{t}}, \circ)$ is unital if and only if L is completely distributive. In this case, the unit is the identity map and

$$[L, L]_{\mathtt{t}} = [L, L]\,,$$

that is, every sup-preserving endomap of L is tight.

A bit of fun: tight endomaps of M_n

Let M_n be the generalized diamond lattice with n atoms (=coatoms).

Theorem. The following are equivalent:

1. f is tight,
2. the image of f has at most two atoms,
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Proposition. There are

$$2 + 2n + 2n^2 + \binom{n}{2} n(n-1) = \frac{1}{2}n^4 - n^3 + \frac{5}{2}n^2 + 2n + 2$$

tight endomaps of M_n.

Negation in $[M_n, M_n]_{\mathtt{t}}$

For x, y, z, w atoms of M_n (with $x \neq z$ and $y \neq w$), let

$$f_{x\mapsto y, z\mapsto w}(t) := \begin{cases} \bot\,, & t = \bot\,, \\ y\,, & t = x\,, \\ w\,, & t = z\,, \\ \top\,, & \text{otherwise}\,. \end{cases}$$

Then

$$(f_{x\mapsto y, z\mapsto w})^* = f_{y\mapsto z, w\mapsto x}\,.$$

(Not the complete history).

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Serre Galois connections (and the double negation construction)

Definition. A Galois connection on a quantale l, r $(Q, *)$ is *Serre* if

- $l \circ r = r \circ l$ and
- $x \backslash l(x) = r(x)/y$, for each $x, y \in Q$.

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Proposition. Let l, r be a Serre Galois connection on $(Q, *)$. Let $j := l \circ r = r \circ l$ and $Q_j = \{\, x \in Q \mid j(x) = x \,\}$. Then j is a nucleus and $(Q_j, *_j)$ is a Frobenius quantale with, as negations, the restrictions of l and r to Q_j.

Proposition. If $(Q_j, *_j)$ is a Frobenius quantale—with ${}^{\perp}(-), (-)^{\perp}$—then $l := {}^{\perp}(-) \circ j$ and $r := {}^{\perp}(-) \circ j$ form a Serre Galois connection, and $j = l \circ r$.

Remark. All of this well-known with units and for Girard quantales. Here without units and for Frobenius quantales.

Frobenius phase quantales

Definition. A phase quantale is of the form $(P(S)_j, \bullet_j)$ where $(P(S), \bullet)$ is the free quantale over a semigroup $(S, \cdot)$ and $j = l \circ r$ for a Serre Galois connection l, r.

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Proposition. Serre Galois connections on $P(S)$ bijectively correspond to binary relations R such that:

1. for all x there exists $Y_x \subseteq S$ such that xRz iff zRy, for each $y \in Y_x$,
2. condition 1. for the converse of R,
3. *associative*: $x \cdot yRz$ if and only if $xRy \cdot z$, for each $x, y, z \in S$.

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Theorem. Every Frobenius quantale is isomorphic to the phase quantale $(P(Q)_j, \bullet_j)$ whose Serre Galois connection is induced by the binary relation

$$xRy \quad \text{iff} \quad x \leq {}^{\perp}y\,.$$

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Frobenius quantales from pregroups

A *pregroup* is an ordered monoid $(M, \leq, \cdot)$ with inverse bijections $l, r : M \longrightarrow M$ such that

$$1 \leq r(x) \cdot x\,, \qquad x \cdot r(x) \leq 1\,, \qquad 1 \leq x \cdot l(x)\,, \qquad l(x) \cdot x \leq 1\,,$$

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R is "representable":

$$x \cdot y \leq 1 \quad \text{iff} \quad x \leq l(y) \quad \text{iff} \quad y \leq r(x)\,.$$

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$Max(A)$ quantales as phase quantales

Let A be an algebra coming with an associative symmetric bilinear map $\langle -, - \rangle$.

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Let A be an algebra coming with an associative symmetric bilinear map $\langle -, - \rangle$.

Then

$$xRy \quad \text{iff} \quad \langle x, y \rangle = 0$$

is associative and yields a (self-adjoint) Serre Galois connection on $P(A)$ and $(P(A)_j, \bullet_j)$ is a Girard quantale. j-closed subspaces are subvector spaces of A.

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Suppose also that there is an involution $(-)^* : A \longrightarrow A$ making $\langle -, - \rangle$ into a sort of inner product:

$$\langle x, x^* \rangle = 0 \quad \text{implies} \quad x = 0\,.$$

For example: A is a $\mathbb{C}^*$-algebra—in which case $(P(A)_j, \bullet_j)$ is called $Max(A)$.

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Theorem. The Girard quantale $(P(A)_j, \bullet_j)$ has a unit if and only if the algebra A has a unit.

$Max(A)$ construction for trace class operators

Until now, $Max(A)$ considered only when A is unital.
In particular when A is the $\mathbb{C}^*$-algebra of matrices over a finite dimensional vector space over $\mathbb{C}$.

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Let H be an infinite dimensional Hilbert space. For a trace class operators $\phi : H \longrightarrow H$, we can define

$$tr(\phi) := \sum_{e \in \mathcal{E}} \langle \phi(e), e \rangle_H \,, \qquad \langle \phi, \psi \rangle := tr(\phi \circ \psi)\,,$$

yielding an associative symmetric pairing and a Serre Galois connection.

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Let $\mathcal{L}^1(H)$ be the ideal of trace class operators. It has no unit, and it is closed under adjoints: $\phi \in \mathcal{L}^1(H)$ implies $\phi^* \in \mathcal{L}^1(H)$. Thus:

Theorem. $Max(\mathcal{L}^1(H))$ is a unitless Frobenius quantale.

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Lemma. A Frobenius quantale $(Q, *)$ is unital if and only if $\bigwedge_{x \in Q} x \backslash x$ is positive.

Theorem. If a Frobenius quantale has no unit, then it cannot be embedded into a unital Frobenius quantale while preserving negations.

A closer look at $[M_n, M_n]_{\mathtt{t}}$

In $[M_n, M_n]_{\mathtt{t}}$ elements of the form $x \backslash x = x^{\perp}/x^{\perp}$ are positive, since they coincide with the same expression computed in $[M_n, M_n]$.

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[Blount, 1999] argues that a residuated partially-ordered semigroup embeds into a residuated partially-ordered monoid if and only if elements of the form $x\backslash x$, x/x are positive.

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Counter-example ! $[M_n, M_n]_{\mathtt{t}}$ shows that the same condition does not suffice for embeddability into unital residuated lattices.

Positive elements via duality

Necessarily, positive elements are not closed under meets in $[M_n, M_n]_{\mathtt{t}}$.

This can also be seen as follows.

Lemma. Sup-preserving closure operators on L dually correspond to complete sublattices of L.

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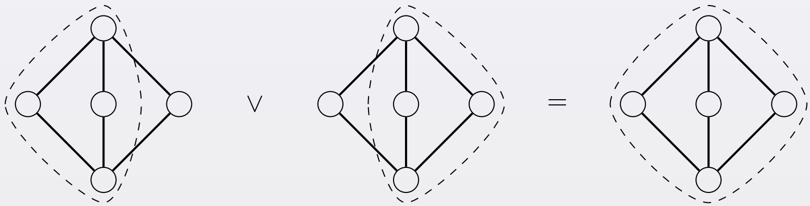
Lemma. Sup-preserving closure operators on L dually correspond to complete sublattices of L.

The join of distributive sublattices is not distributive:



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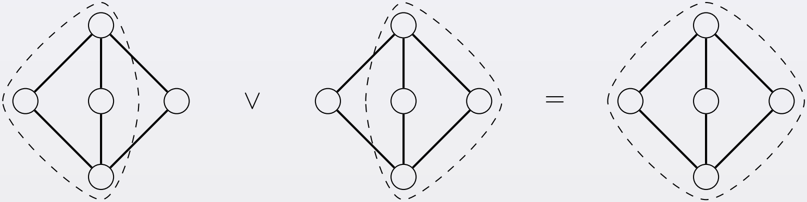

Thus, for two tight closure operators j_1, j_2, we have in $[M_n, M_n]_{\mathtt{t}}$

$$j_1 \wedge j_2 = id_{M_n}$$

and, within $[M_n, M_n]_{\mathtt{t}}$,

$$j_1 \wedge j_2 = (id_{M_n})^{\wedge\vee} = \bot \, .$$

Obrigado !

Perguntas ?

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