Unified inverse correspondence for DLE-Logics

Mattia Panettiere

Vrije Universiteit, Amsterdam

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The problem

Correspondence

$$\Box p \to \Box\Box p$$

$$\rightsquigarrow$$

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Inverse correspondence

$$\exists y \big[Rxy \ \& \ \forall z \big(Ryz \Rightarrow \exists w (Rzw \ \& \ Rwx \ \& \ Rxw) \big) \big]$$

$$\rightsquigarrow$$

$$p \wedge \Box(\Diamond p \to \Box q) \leq \Diamond\Box q$$

(Partial) solution in the classical setting

Kracht formulae

$$\chi(x_0) = (\forall x_1 \rhd x_{i_1}) \dots (\forall x_n \rhd x_{i_n})(Q_1 y_1 \rhd z_{j_1}) \dots (Q_m y_m \rhd z_{j_m})\beta,$$

where

- Q_i are quantifiers,
- x_0 is the only free variable,
- β is a DNF of atoms involving variables in $\{x_1, \dots, x_n\}$

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Sahlqvist formulae $\leftrightsquigarrow$ Kracht formulae

Generalizing to inductive - Problems

$$p \wedge \Box(\Diamond p \rightarrow \Box q) \leq \Diamond\Box\Box q$$

has FO correspondent

$$\forall x \exists y (xRy \ \& \ \forall z (yR^2 z \Rightarrow \exists w (wRz \ \& \ wRx \ \& \ xRw)))$$

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$$\forall x (\exists y \rhd x)(\forall z_1 \rhd y)(\forall z \rhd z_1)(\exists w \blacktriangleright z)(wRx \ \& \ xRw)$$
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Language for unified correspondence

A (D)LE language $\mathcal{F}(\mathcal{F}, \mathcal{G})$ is extended to

$$\varphi ::= \mathbf{j} \mid \mathbf{m} \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\varphi, \ldots, \varphi) \mid g(\varphi, \ldots, \varphi),$$

where $p \in \mathrm{AtProp}$, $\mathbf{j} \in \mathit{NOM}$, $\mathbf{m} \in \mathit{CONOM}$, and $f \in \mathcal{F}^*, g \in \mathcal{G}^*$.

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$$\begin{array}{rcl} \text{Algebraic interpretation} & \rightsquigarrow & \text{Agnostic wrt. semantics} \\ \text{Arbitrary } (\mathcal{F}, \mathcal{G}) & \rightsquigarrow & \text{Agnostic wrt. signature} \end{array}$$

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$$\begin{array}{rcl} NL &=& NOM \quad \cup \quad \{\lambda(\mathbf{m}) : \mathbf{m} \in CONOM\}, \\ CNL &=& CONOM \quad \cup \quad \{\kappa(\mathbf{j}) : \mathbf{j} \in NOM\}. \end{array}$$

We write $\rho\mathbf{j}$ to denote $\kappa(\mathbf{j})$, and $\rho\mathbf{m}$ for $\lambda(\mathbf{m})$.

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Lemma

For all $\mathbf{j}$, $\mathbf{m}$, *and* φ,

$$\mathbf{j} \leq \varphi \quad \textit{iff} \quad \varphi \nleq \kappa(\mathbf{j}) \qquad\qquad \varphi \leq \mathbf{m} \quad \textit{iff} \quad \lambda(\mathbf{m}) \nleq \varphi$$

Flat and restricting inequalities

In the classical setting FO-atomic are equivalent to inequalities

xRy	$x \leq \Diamond y \quad \text{iff} \quad \Box y^c \leq x^c$ $y \leq \blacklozenge x \quad \text{iff} \quad \blacksquare x^c \leq y^c$
$x = y$	$x \leq y \quad \text{iff} \quad y^c \leq x^c \quad \text{iff} \quad x \nleq y^c$
$x = y = z$	$x \to y^c \leq z^c \quad \text{iff} \quad x \leq y \succ z^c$.

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xRy	$x \leq \Diamond y$ iff $\Box y^c \leq x^c$ $y \leq \blacklozenge x$ iff $\blacksquare x^c \leq y^c$
$x = y$	$x \leq y$ iff $y^c \leq x^c$ iff $x \not\leq y^c$
$x = y = z$	$x \to y^c \leq z^c$ iff $x \leq y \mathrel{>\!\!-} z^c$.

Definition (Restricting inequality)

Restricting inequalities are inequalities of shape

$$\mathbf{i} \leq f(\bar{\mathbf{j}}, \overline{\mathbf{m}}), \qquad g(\overline{\mathbf{m}}, \bar{\mathbf{j}}) \leq \mathbf{n}, \qquad \mathbf{i} \leq \mathbf{h}, \qquad \mathbf{o} \leq \mathbf{n}.$$

Restricted quantifiers

$$
\begin{aligned}
(\forall \bar{\mathbf{i}}, \overline{\mathbf{n}} \rhd_f \mathbf{j})\beta &\equiv (\forall \bar{\mathbf{i}}, \overline{\mathbf{n}})(\quad \mathbf{j} \leq f(\bar{\mathbf{i}}, \overline{\mathbf{n}}) \quad \Rightarrow \quad \beta \quad), \\
(\forall \bar{\mathbf{i}}, \overline{\mathbf{n}} \rhd_g \mathbf{m})\beta &\equiv (\forall \bar{\mathbf{i}}, \overline{\mathbf{n}})(\quad g(\bar{\mathbf{i}}, \overline{\mathbf{n}}) \leq \mathbf{m} \quad \Rightarrow \quad \beta \quad), \\
(\exists \bar{\mathbf{i}}, \overline{\mathbf{n}} \rhd_f \mathbf{j})\beta &\equiv (\exists \bar{\mathbf{i}}, \overline{\mathbf{n}})(\quad \mathbf{j} \leq f(\bar{\mathbf{i}}, \overline{\mathbf{n}}) \quad \& \quad \beta \quad), \\
(\exists \bar{\mathbf{i}}, \overline{\mathbf{n}} \rhd_g \mathbf{m})\beta &\equiv (\exists \bar{\mathbf{i}}, \overline{\mathbf{n}})(\quad g(\bar{\mathbf{i}}, \overline{\mathbf{n}}) \leq \mathbf{m} \quad \& \quad \beta \quad),
\end{aligned}
$$

Flattification

$$\begin{array}{rcl}
\mathbf{j} \leq f(\overline{\alpha}, \overline{\beta}) & \text{iff} & (\exists \bar{\mathbf{i}}, \overline{\mathbf{n}} \rhd_f \mathbf{j})(\bigotimes \mathbf{i} \leq \alpha \ \ \& \ \ \bigotimes \beta \leq \mathbf{n}) \\
f(\overline{\alpha}, \overline{\beta}) \leq \mathbf{m} & \text{iff} & (\forall \bar{\mathbf{i}}, \overline{\mathbf{n}} \rhd_f \lambda\mathbf{m})(\bigparr \alpha \leq \kappa\mathbf{i} \ \ \parr \ \ \bigparr \lambda\mathbf{n} \leq \beta) \\
\mathbf{j} \leq g(\overline{\alpha}, \overline{\beta}) & \text{iff} & (\forall \overline{\mathbf{n}}, \bar{\mathbf{i}} \rhd_g \kappa\mathbf{j})(\bigparr \lambda\mathbf{n} \leq \alpha \ \ \parr \ \ \bigparr \beta \leq \kappa\mathbf{i}) \\
g(\overline{\alpha}, \overline{\beta}) \leq \mathbf{m} & \text{iff} & (\exists \overline{\mathbf{n}}, \bar{\mathbf{i}} \rhd_g \mathbf{m})(\bigotimes \alpha \leq \mathbf{n} \ \ \& \ \ \bigotimes \mathbf{i} \leq \beta)
\end{array}$$

Conclusions

Main contributions:

- Sahlqvist $\rightsquigarrow$ very simple Sahlqvist in $\mathcal{L}^* \supseteq$ inductive
- Boolean $\rightsquigarrow$ distributive
- semantic and signature-agnostic

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Thank you!