Lattice-ordered groups and monoids

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Outline

1. Background on ℓ-groups.
2. Removing inverses: distributive ℓ-monoids.
3. Removing inverses in 3 cases: abelian ℓ-groups, semilinear ℓ-groups, ℓ-groups.

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3. Removing inverses in 3 cases: abelian ℓ-groups, semilinear ℓ-groups, ℓ-groups.
4. Axiomatization of semilinear distributive ℓ-monoids.
5. A nice proof-theoretic/syntactic system for ℓ-groups via DLMs.
6. Connections to total orders on free groups and free monoids.

From distributive ℓ-monoids to ℓ-groups, and back again, A. Colacito, N. Galatos, G. Metcalfe, S. Santchi, Journal of Algebra 601 (2022), 129—148.

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Examples:

- $(\mathbb{Z}, min, max, +, -, 0)$, $(\mathbb{Q}, min, max, +, -, 0)$, $(\mathbb{R}, min, max, +, -, 0)$.
- $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Q} \times \mathbb{R}$. (direct product)
- $\mathbb{Z} \overrightarrow{\times} \mathbb{R}$ (lexicographic order).
- The order-preserving permutations (aka automorphisms) $\mathbf{Aut}(C, \leq)$ on a totally-ordered set $(C, \leq)$, under functional composition and pointwise order. For example, the *symmetric* ℓ-groups: $\mathbf{Aut}(\mathbf{n})$, $\mathbf{Aut}(\mathbb{N})$, $\mathbf{Aut}(\mathbb{Z})$, $\mathbf{Aut}(\mathbb{R})$.

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Holland's embedding theorem. Every ℓ-group can be embedded in a symmetric ℓ-group: $\mathbf{G} \hookrightarrow \mathbf{Aut}(\mathbf{C})$, for some chain $\mathbf{C}$.

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Theorem. (FMP) DLM is generated by $\{\mathbf{End}(\mathbf{C}) : \mathbf{C}$ is a finite chain$\}$. Equivalently, an inverse-free equation fails in a DML then it fails in some $\mathbf{End}(\mathbf{C})$, where $\mathbf{C}$ is finite.
Here, $\mathbf{End}(\mathbf{C})$ is the DLM of order-preserving functions from $\mathbf{C}$ to $\mathbf{C}$.

Examples of DLMs; the Finite Model Property

Examples. The DLMs coming from ℓ-groups (e.g., $\mathbf{Aut}(\mathbf{C})$): all infinite (or trivial).

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Due to the various distributivities every equation can be written as $s_1 \wedge \cdots \wedge s_n \leq t_1 \vee \cdots \vee t_m$, where s_i, t_j are monoid/group terms.

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So, there is $p \in C$ such that $s(f_1, \ldots, f_n)(p) \neq t(f_1, \ldots, f_n)(p)$ ($(f \circ g)(p) \neq (g \circ f)(p)$).

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Let $\mathbf{C}'$ be the finite subchain of $\mathbf{C}$ containing all the points involved in the evaluation of the failure. (eg, $p, g(p), f(g(p)), f(p), g(f(p))$)

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(Eg, $p \overset{f}{\mapsto} f(p)$, $g(p) \overset{f}{\mapsto} f(g(p))$, $p \overset{g}{\mapsto} g(p)$, $f(p) \overset{g}{\mapsto} g(f(p))$.) Called a *diagram*.

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For the FMP: we extend the f_i''s to $\mathbf{End}(\mathbf{C}')$ (in a non-injective way).

Inverse-free reducts

Lemma. If an inverse-free equation fails in an $\mathbf{End}(\mathbf{C})$, where $\mathbf{C}$ is a finite chain, then it fails in $\mathbf{Aut_m}(\mathbb{Q})$, the inverse-free reduct of $\mathbf{Aut}(\mathbb{Q})$.

Proof-idea: As before, we obtain a *diagram*, as before. For example:

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If the partial maps in the diagram are all injective, then using the density of $\mathbb{Q}$ we embed the finite chain $\mathbf{C}'$ in $\mathbb{Q}$ and extend these partial injections to total bijections of $\mathbb{Q}$, which still show the failure at $p \in \mathbf{C}' \subseteq \mathbb{Q}$.

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To create $\overline{\mathbf{C}}$ we duplicate points $q \in C$ where $f(p_1) = f(p_2) = q$ for some f to get $\overline{f}(p_1) = q_1$ and $\overline{f}(p_2) = q_2$.

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If the partial maps in the diagram are all injective, then using the density of $\mathbb{Q}$ we embed the finite chain $\mathbf{C}'$ in $\mathbb{Q}$ and extend these partial injections to total bijections of $\mathbb{Q}$, which still show the failure at $p \in \mathbf{C}' \subseteq \mathbb{Q}$.

If the partial maps in the diagram are not injective, then we duplicate the points of $\mathbf{C}'$, so as to construct a big enough, but still finite, chain $\overline{\mathbf{C}}$ and *partial injections* which still show the failure.

To create $\overline{\mathbf{C}}$ we duplicate points $q \in C$ where $f(p_1) = f(p_2) = q$ for some f to get $\overline{f}(p_1) = q_1$ and $\overline{f}(p_2) = q_2$. Unfortunately, if we have $g(q) = r$, then this creates the further problem of $g(q_1) = g(q_2) = r$, which needs to be further resolved.

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Consider $\mathbf{End}(\mathbf{2})$, where $\mathbf{2} = \langle \{0, 1\}, \leq \rangle$ is the two-element chain.

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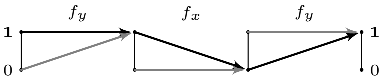
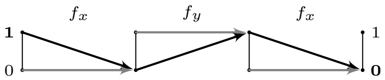
$$(1)f_{yxy} = (((1)f_y)f_x)f_y = 1 > 0 = (((1)f_x)f_y)f_x = (1)f_{xyx}.$$





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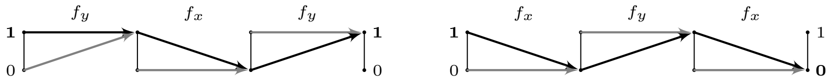
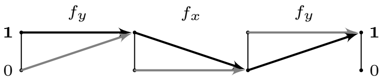


To construct the chain $\overline{C}$, we consider all initial segments of the terms yxy and xyx and the paths created by their applications to $p = 1$: $\varepsilon = (1)$, $y = (1,1)$, $yx = (1,1,0)$, $yxy = (1,1,0,1)$, $x = (1,0)$, $xy = (1,0,1)$, and $xyx = (1,0,1,0)$. We order these paths with the reverse lexicographic order:

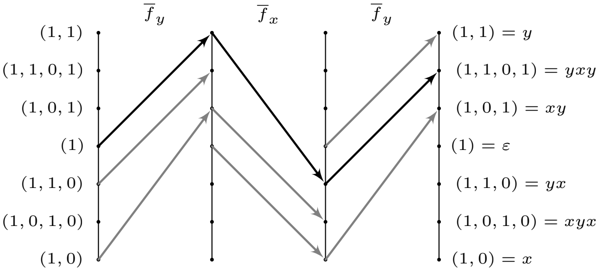
$$(1,0) < (1,0,1,0) < (1,1,0) < (1) < (1,0,1) < (1,1,0,1) < (1,1),$$

where the first three elements serve as copies of 0 and the last four as copies of 1.

Example (2)



becomes:



Where $(p, p_1, \ldots, p_n)\overline{f} = (p, p_1, \ldots, p_n, (p_n)f)$, if the latter is a path.

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Corollary. To decide (inverse-including) equations in ℓ-groups, we only need to be able to decide (inverse-free) equations in DLMs.

Hybrid system. Given an ℓ-group equation we apply (upward) instances of the density rule until we obtain an inverse-free equation. Then we continue in the system $\mathbf{DLM}$.

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Adding a further axiom to a transitivity-free system typically breaks cut-freeness. So, we *inject some transitivity* into the the distributivity of multiplication over meet

$$xz \wedge xw \leq x(z \wedge w)$$

to first get its linearized version

$$xz \wedge yw \leq x(z \wedge w) \vee y(z \wedge w)$$

and then the quasiequation

$$\frac{x(z \wedge w) \leq c \quad y(z \wedge w) \leq c}{xz \wedge yw \leq c}$$

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Theorem. [G. - Jipsen] $\mathbf{DLM}$ admits (two-sorted) relational semantics (residuated frames).

Decidable and cut-free systems

To analyze proofs better, it helps if our derivation system does not include the rule of transitivity/cut: $a \leq b$ and $b \leq c$ implies $a \leq c$.

[G. - Metcalfe, 2016] gives a derivation system for ℓ-groups without the rule of transitivity (and no other rule, where an unexpected term like b appears during a proof search; i.e., the system is *analytic*).

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Shortcoming: Neither system allows for a good duality theory, as provided by residuated frames **[G. - Jipsen]**. This is because the ℓ-group axioms are high in the substructural hierarchy **[Ciabbatoni - G. - Terui]**.

Inverse-free reducts of representable/semilinear

Theorem. The inverse-free subreducts of semilinear ℓ-groups are not the whole variety of semilinear (subdirect product of chains) DLMs.

Proof idea: We define the terms

$F = x_1x_2x_3 \wedge x_5x_4x_6 \wedge x_9x_7x_8$, $G = x_1x_4x_7 \vee x_5x_2x_8 \vee x_9x_6x_3$,
$F' = x_1x_3x_2 \wedge x_5x_6x_4 \wedge x_9x_8x_7$, $G' = x_1x_7x_4 \vee x_5x_8x_2 \vee x_9x_3x_6$.

We prove that $F \wedge F' \leq G \vee G'$ holds in all totally ordered groups. This is done by presenting a derivation in the system of **[G-M 2016]** expanded by the *gen-cycle* quasiequation ($1 \leq s \vee gh \;\Rightarrow\; 1 \leq s \vee hg$), which holds in the free representable ℓ-group.

We also show that $F \wedge F' \leq G \vee G'$ fails in a *commutative* totally ordered monoid. (Note that in the commutative case $F = F'$ and $G = G'$.)

Conjecture. The inverse-free subreducts of representatble ℓ-groups do not form a finitely axiomatizable variety (over the semilinear (distributive) ℓ-monoids).

We should first axiomatize the variety of semilinear DLMs.

Semilinear DLMs

Theorem. Semilinear DLMs are axiomatized by the equation

$$z_1xz_2 \wedge w_1yw_2 \leq z_1yz_2 \vee w_1xw_2.$$

Note that it implies $ee(yx) \wedge yxe \leq ex(yx) \vee yee$, namely $yx \leq xyx \vee y$.

The proof uses ideas from **[G. - Horčík]**, where Holland-type theorems are established for residuated lattices and semilattice-ordered monoids. Also, it uses ideas from **[Melier]**.

For an DLM $\mathbf{M}$ and prime ideal I, we get a congruence:

$$a \sim_I b \text{ iff for all } z, w \in M,\ zaw \in I \text{ iff } zbw \in I.$$

The quotient $\mathbf{M}/I$ is a chain iff

$$z_1xz_2 \in I \text{ and } w_1yw_2 \in I \text{ implies } z_1yz_2 \in I \text{ or } w_1xw_2 \in I.$$

Proof sketch: (It works even for non-distributive and non-lattice-ordered)

1. Relatively maximal ideals are prime and they produce linear quotients and
2. we have enough relatively maximal to separate points.

We obtain a calculus for the semilinear case by transforming the above equation into the quasiequation

$$\frac{z_1yz_2 \leq c \quad w_1xw_2 \leq c}{z_1xz_2 \wedge w_1yw_2 \leq c}$$

Also, adding commutativity/exchange gives a calculus for commutative DLMs.

Orders on the free group

Fact. The lattice order of any ℓ-group is the intersection of all of its total-order extensions that are *right orders* (orders compatible with right multiplication).

Fact. Every total right order on a group is determined by its positive cone.

Fact. Total orders on the *free abelian group* on two generators are in bijective correspondence with lines through the origin with irrational slope together with (counted twice) lines through the origin with rational slope.

Theorem [Colacito - Metcalfe] The following are equivalent

1. $\{t_1, \ldots, , t_n\}$ extends to the positive cone of a right order on the free group over X.
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2. $\{s_1 < t_1, \ldots, s_n < t_n\}$ extends to a right order on the *free group* over X.
3. $\mathsf{LG} \not\models e \leq s_1^{-1}t_1 \vee \cdots \vee s_n^{-1}t_n$.
4. $\not\models_{\mathsf{LG}} y_1s_1 \wedge \cdots \wedge y_ns_n \leq y_1t_1 \vee \cdots \vee y_nt_n$. (The variables $y_1, y_2, \ldots, y_n$ are fresh.)
5. $\not\models_{\mathsf{DLM}} y_1s_1 \wedge \cdots \wedge y_ns_n \leq y_1t_1 \vee \cdots \vee y_nt_n$.

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Thank you!! ☺