The Dependence-Problem in Varieties of Modal Semilattices

Naomi Tokuda

University of Bern

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Initial Motivation

De Jongh and Chagrova[1] call formulas $\varphi_1, \ldots, \varphi_n$ of intuitionistic logic dependent whenever for some formula ψ in the variables $y_1, \ldots, y_n$,

$$\vdash_{\mathsf{IPC}} \psi(\varphi_1, \ldots, \varphi_n), \text{ but } \nvdash_{\mathsf{IPC}} \psi(y_1, \ldots, y_n).$$

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Using Pitts' constructive proof of uniform interpolation for IPC, they also showed that the dependence of finitely many formulas is decidable.

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Motivation for Considering Modal Semilattices

For modal logic K, the proof for IPC cannot be applied, but it follows from results on uniform interpolation in description logic by Lutz and Wolter[2], that the same question for K is decidable.

[2] C. Lutz and F. Wolter, "Foundations for uniform interpolation and forgetting in expressive description logics", Proc. IJCAI 2011, 989–995 (2011).

[3] S. Kikot et al., "Kripke completeness of strictly positive modal logics over meet-semilattices with operators", The Journal of Symbolic Logic **84**, 533–588 (2019).

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In this talk we consider two of these fragments algebraically. Structures like these are considered for example by Kikot et al[3]. These fragments are also relevant when considering weaker description logics.

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Preliminaries

Let $\mathcal{L}$ be an algebraic language and let $\mathcal{V}$ be a variety (equational class) of $\mathcal{L}$-algebras. By $\mathrm{Tm}(\bar{x})$ and $\mathrm{Eq}(\bar{x})$, we denote the sets of $\mathcal{L}$-terms and $\mathcal{L}$-equations over the set of variables $\bar{x}$, respectively.

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$$\mathcal{V} \models s \approx t \quad :\Longleftrightarrow \quad f(s) = f(t) \text{ for all } \mathbf{A} \in \mathcal{V} \text{ and all homomorphisms } f \text{ from } \mathbf{Tm}(\bar{x}) \text{ to } \mathbf{A}.$$

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For $\Sigma \subseteq \mathrm{Eq}(\bar{x})$, we write $\mathcal{V} \models \Sigma$, whenever $\mathcal{V} \models s \approx t$ for all $s \approx t \in \Sigma$.

Dependence

Let $t_1, \dots, t_n$ be $\mathcal{L}$-terms in the variables $\bar{x} = \{x_1, \dots, x_m\}$ and let $\mathcal{V}$ be a variety of $\mathcal{L}$-algebras.

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Then $t_1, \ldots, t_n$ are called $\mathcal{V}$-dependent if there is an equation $\varepsilon(y_1, \ldots, y_n)$ such that

$$\mathcal{V} \vDash \varepsilon(t_1, \ldots, t_n) \quad \text{but} \quad \mathcal{V} \nvDash \varepsilon(y_1, \ldots, y_n).$$

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The problem of deciding whether any finite number of $\mathcal{L}$-terms are $\mathcal{V}$-dependent is called the dependence problem for $\mathcal{V}$.

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$\mathcal{V}$-dependence corresponds to a special case of a notion of dependence studied by Marczewski[4] and others.

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$\mathcal{V}$-Refuting Sets and Dependence

A set $\Delta \subseteq \mathrm{Eq}(\bar{y})$ is called $\mathcal{V}$-refuting for $\bar{y}$, if the following hold:

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1. For each $\delta \in \Delta$, $\mathcal{V} \not\models \delta$;
2. For any equation $\varepsilon(\bar{y})$, such that $\mathcal{V} \not\models \varepsilon$, and any substitution $\sigma \colon \mathbf{Tm}(\bar{y}) \to \mathbf{Tm}(\omega)$,

$$\mathcal{V} \models \sigma(\varepsilon) \quad \Longrightarrow \quad \mathcal{V} \models \sigma(\delta) \text{ for some } \delta \in \Delta,$$

where σ is extended to equations by setting $\sigma(s \approx t) = \sigma(s) \approx \sigma(t)$.

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Lemma

For any $\mathcal{V}$-refuting set $\Delta(\bar{y})$ for $\bar{y} = \{y_1, \ldots, y_n\}$, the terms $t_1, \ldots, t_n \in \mathrm{Tm}(\bar{x})$ are $\mathcal{V}$-dependent if and only if $\mathcal{V} \vDash \delta(t_1, \ldots, t_n)$ for some $\delta \in \Delta$.

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Thus, for varieties that have a decidable equational theory and for which a finite $\mathcal{V}$-refuting set for any finite $\bar{y}$ can be constructed, the dependence problem is decidable.

Example[5]

Note that for varieties of algebras with an order, for the dependence problem, we can consider inequations instead of equations.

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We define $[n] := \{1, \dots, n\}$. Let us consider $\mathcal{L}at$, the variety of all lattices, and let

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$$\Delta_n \coloneqq \Big\{ y_i \leq \bigvee_{j \in [n] \setminus \{i\}} y_j \mid i \in [n] \Big\} \cup \Big\{ \bigwedge_{j \in [n] \setminus \{i\}} y_j \leq y_i \mid i \in [n] \Big\}.$$

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We can show that Δ_n is a $\mathcal{L}at$-refuting set for $\{y_1, \ldots, y_n\}$ and thus, the dependence problem for $\mathcal{L}at$ is decidable.

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Modal Join-Semilattices

Let $\mathcal{MJS}$ be the variety of $\langle \vee, \Box \rangle$-algebras $\langle A, \vee, \Box \rangle$ such that $\langle A, \vee \rangle$ is a semilattice and for all $a, b \in A$,

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For notational convenience, we write for a term s, $\varnothing \vee s = s \vee \varnothing = s$.

Lemma

The following set of $\mathcal{MJS}$-inequations in $\bar{y}$ is $\mathcal{MJS}$-refuting for $\bar{y}$:

$$\Delta_{\bar{y}} := \{y \leq s \mid y \in \bar{y} \text{ and } s \neq s_1 \vee y \vee s_2 \text{ for } s_1, s_2 \in \mathrm{Tm}(\bar{y}) \cup \{\varnothing\}\} \\ \cup \{\Box^k y \leq y' \mid y, y' \in \bar{y} \text{ and } k > 0\}.$$

Proof Idea by Example

We prove this by giving a procedure that for any inequation ε not valid in $\mathcal{MJS}$ yields a finite set $\Delta_\varepsilon \subseteq \Delta_{\bar{y}}$ such that for any substitution σ,

$$\mathcal{MJS} \vDash \sigma(\varepsilon) \quad \Longrightarrow \quad \mathcal{MJS} \vDash \sigma(\delta) \text{ for some } \delta \in \Delta_\varepsilon.$$

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Now, either $\mathcal{MJS} \vDash \sigma(y_1) \leq \sigma(y_3)$ or $\mathcal{MJS} \vDash \sigma(y_1) \leq s_2$ where $\sigma(y_2) = s_1 \vee \Box s_2 \vee s_3$ for $s_2 \in \mathrm{Tm}(\bar{y})$ and $s_1, s_3 \in \mathrm{Tm}(\bar{y}) \cup \{\varnothing\}$. In the second case, we get $\mathcal{MJS} \vDash \Box\sigma(y_1) \leq \sigma(y_2)$.

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To show the idea, we consider an example. Let $\varepsilon = \Box^2 y_1 \leq \Box(y_2 \vee \Box y_3)$.

Clearly, $\mathcal{MJS} \not\models \varepsilon$ and $\mathcal{MJS} \not\models \Box y_1 \leq y_2 \vee \Box y_3$.

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Also, $\mathcal{MJS} \not\models \Box y_1 \leq y_2$ and $\mathcal{MJS} \not\models y_1 \leq y_3$. Thus,

$$\Delta_\varepsilon = \{\Box y_1 \leq y_2, y_1 \leq y_3\}.$$

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Theorem

Let $t_1, \ldots, t_n \in \mathrm{Tm}(\bar{x})$ and let $\bar{y} = \{y_1, \ldots, y_n\}$. Then $t_1, \ldots, t_n$ are $\mathcal{MJS}$-dependent if and only if there is an inequation $\delta \in \Delta_{\bar{y}}^d$ such that

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where $d := \max\{\mathrm{md}(t_1), \ldots, \mathrm{md}(t_n)\}$ and $\Delta_{\bar{y}}^d := \{\delta \in \Delta_{\bar{y}} \mid \mathrm{md}(\delta) \leq d\}$.

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Corollary

The dependence problem for $\mathcal{MJS}$ is decidable.

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$\mathbf{F}(\bar{x})$, the free $\mathcal{MMS}$-algebra over $m > 0$ generators is isomorphic to the following $\mathcal{MMS}$-algebra:

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where $\mathcal{P}_{fin}(\mathbb{N})$ is the set of all finite subsets of $\mathbb{N}$, and $\cup, \Box$ are defined component-wise with $\Box \{a_1, \dots, a_k\} := \{a_1 + 1, \dots, a_k + 1\}$ for $a_1, \dots, a_k \in \mathbb{N}$.

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Ongoing/Future Research and Open Problems

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- Are there varieties with a decidable equational theory, for which the dependence problem is undecidable?

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