On varieties of residuated po-magmas and the structure of finite involutive po-semilattices

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TACL 2022: Topology, Algebra and Categories in Logic
Coimbra, June 20-24, 2022

Partially ordered algebras

A **po-algebra** is a partially ordered set with operations that are either order-preserving **or order-reversing** in each argument.

A **variety of po-algebras** is a class of similar po-algebras defined by equations **or inequations** [Pigozzi 2004].

A **residuated partially ordered magma** or **rpo-magma** $\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a partially-ordered set $(A, \leq)$ with a binary operation $\cdot$ and two **residuals** that satisfy for all $x, y, z \in A$

$$\text{(res)} \qquad xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$$

The operation $x \cdot y$ is usually written xy.

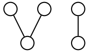
Residuated po-magmas are a po-variety

Residuation ensures that x/y and $y\backslash x$ are order-preserving in the numerator (x position) and **order-reversing** in the denominator.

xy is order-preserving in both arguments.

(res) is equivalent to $x \le xy/y$, $(z/y)y \le z$, $y \le x\backslash xy$, $x(x\backslash z) \le z$ hence rpo-magmas are a variety of po-algebras.

Although rpo-magmas are very general, (res) imposes restrictions on the posets that can occur.

E.g. could  be the poset of a rpo-magma?

Lemma

For rpo-magmas, if $a, b \leq c$ *then* $(a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq a, b$.

Proof.

Assume $a \leq c$. Then $a/(a\backslash c) \leq c/(a\backslash c)$

$$\Longrightarrow \quad (c/(a\backslash c))\backslash b \leq (a/(a\backslash c))\backslash b$$

$$\Longleftrightarrow \quad (a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq b$$

Assume $b \leq c$. Then $a\backslash c \leq a\backslash c$

$$\Longleftrightarrow \quad a(a\backslash c) \leq c$$

$$\Longleftrightarrow \quad a \leq c/(a\backslash c)$$

$$\Longrightarrow \quad (c/(a\backslash c))\backslash b \leq a\backslash b \leq a\backslash c$$

$$\Longrightarrow \quad a/(a\backslash c) \leq a/((c/(a\backslash c))\backslash b)$$

$$\Longleftrightarrow \quad (a/(a\backslash c))((c/(a\backslash c))\backslash b) \leq a$$

Finite rpo-magmas have bounded components

Lemma

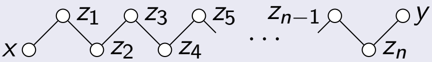
In any rpo-magma, if $d \leq a, b$ then
$a, b \leq d/(((a\backslash d)/(a\backslash(d\backslash d)))((d\backslash d)/(a\backslash(d\backslash d)))\backslash(b\backslash d)))$

Theorem

In an rpo-magma every connected component of $\leq$ is up-directed and down-directed, hence for **finite** *rpo-magmas every connected component is* **bounded***.*

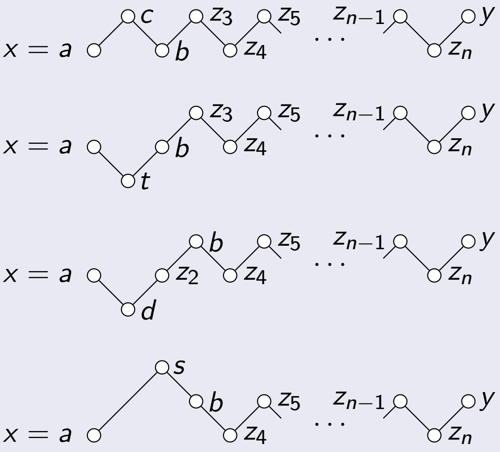
Proof.

Two elements x, y in a poset are connected iff there exists a zigzag



We need to find an upper and a lower bound of x, y.

Proof (continued).



Continue by induction to get an upper and lower bound in n steps.

What posets are possible for ipo-magmas?

Theorem

The equivalence relation on a poset that has each connected component as an equivalence class is a congruence on a rpo-magma, and the quotient algebra is a **quasigroup** *with the discrete order (i.e. $\leq$ is the equality relation).*

Conversely, from any group or quasigroup Q and a pair-wise disjoint family of **bounded** *posets A_q for $q \in Q$, one can construct an rpo-magma with poset $\bigcup_{q\in Q} A_q$.*

E.g. for a group Q and $x_p \in A_p$, $y_q \in A_q$ define

$$x_p \cdot y_q = \bot_{pq}, \qquad x_p \backslash y_q = \top_{p^{-1}q}, \qquad x_p / y_q = \top_{pq^{-1}}.$$

Residuated po-semigroups

A **rpo-semigroup** or **Lambek algebra** is a rpo-magma where $\cdot$ is associative.

Note: If the order is an antichain then a rpo-semigroup is a **group**.

A **unital** rpo-magma has a constant 1 such that $x1 = x = 1x$, and a **rpo-monoid** is a unital rpo-semigroup $(A, \leq, \cdot, \sim, -, 1)$.

A **residuated lattice-ordered magma** $(A, \wedge, \vee, \cdot, \backslash, /)$ (or **rℓ-magma** for short) is a rpo-magma for which the partial order is a lattice order.

A $r\ell$-monoid is more commonly called a **residuated lattice** or **unital quantale** (if it is a complete lattice).

Involutive po-magmas

An **involutive po-magma** or **ipo-magma** $(A, \leq, \cdot, \sim, -)$ is a poset $(A, \leq)$ with a binary operation $\cdot$, two unary **order-reversing operations** $\sim, -$ that are an **involutive pair**: $\sim\!-x = x = -\!\sim\!x$, and for all $x, y, z \in A$

$$\text{(ires)} \qquad xy \leq z \iff x \leq -(y \cdot \sim z) \iff y \leq \sim(-z \cdot x).$$

It follows that ipo-magmas are **rpo-magmas**.

Hence $\cdot$ is order-preserving and ipo-magmas are a **po-variety**.

Models of cardinality $n =$	1	2	3	4	5	6
rpo-magmas	1	3	28	1200		
ipo-magmas	1	4	12	67	314	3029

Involutive po-magmas

A convenient **equivalent** formulation of (ires):

$$\text{(rotate)} \quad xy \le z \iff y \cdot {\sim}z \le {\sim}x \iff -z \cdot x \le -y.$$

The po-variety of ipo-monoids includes **all partially ordered groups** where ${\sim}x = -x = x^{-1}$.

Lemma

Let $\mathbf{A} = (A, \le, \cdot, \sim, -)$ be a poset with a binary operation and two unary operations.

1. *If $\cdot$ is idempotent (i.e. $xx = x$) and $\mathbf{A}$ satisfies* (rotate) *then $\mathbf{A}$ is an ipo-magma.*
2. *If an ipo-magma is idempotent or unital, and $\cdot$ is commutative then ${\sim}x = -x$.*

Involutive po-semilattices

An **ipo-semilattice** $(A, \leq, \cdot, -)$ is an ipo-magma where $\cdot$ is associative, commutative and idempotent.

In an ipo-semilattice there is another partial order $\sqsubseteq$ called the **multiplicative order**, defined by $x \sqsubseteq y \iff xy = x$.

Examples of ipo-semilattices: Boolean algebras $(A, \leq, \cdot, -)$, where join is $-(-x \cdot -y)$.

They form a **po-subvariety** defined by $x \cdot -x \leq y \cdot -y$.

More generally, ipo-semilattices can be visualized by the two Hasse diagrams for $\leq$, $\sqsubseteq$

Visualizing ipo-semilattices

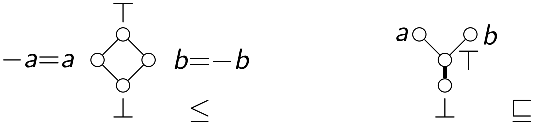


Figure: Partial order and multiplicative order of the smallest ipo-semilattice that does not have an identity element.

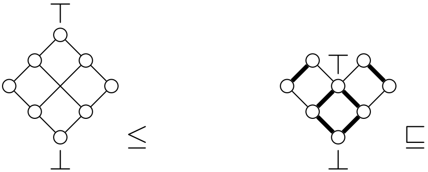


Figure: Smallest ipo-semilattice that is not lattice-ordered.

Unital involutive po-semilattices

An element t in an ipo-semilattice is the **multiplicative identity** iff t is the top element in the multiplicative order.

Hence an ipo-semilattice is **unital** if and only if the multiplicative order has a **top element**.

Sugihara monoid reducts without $\wedge, \vee$ are unital ipo-semilattices.

For finite commutative idempotent involutive residuated lattices (i.e. finite unital iℓ-semilattices) a structural description has been given by [J., Tuyt, Valota 2021].

Structural description for ipo-semilattices

We give a description of ipo-semilattices based on **Płonka sums of Boolean algebras**.

Similar methods are used by Jenei [2022] to describe the structure of **even and odd involutive commutative residuated chains**.

Inspired by a duality for involutive bisemilattices by Bonzio, Loi, Peruzzi [2019], we give a more compact dual description of finite ipo-semilattices based on **semilattice direct systems of partial maps between sets**.

Lemma 1

Let A be a **residuated po-semilattice** and let $x, y \in A$ such that $x\backslash x = y\backslash y$. Then

1. $x \sqsubseteq y \iff x \leq y$,
2. $x\backslash x = xy\backslash xy$,
3. if $y\backslash y = z\backslash z$ then $x\backslash x = yz\backslash yz$, and
4. if $y \sqsubseteq z \sqsubseteq x\backslash x$ then $x\backslash x = z\backslash z$.

Defining an Equivalence Relation

Define an equivalence relation $\equiv$ on A by $x \equiv y \iff x\backslash x = y\backslash y$. Part (1) of the previous lemma shows that the partial order $\leq$ and the semilattice order $\sqsubseteq$ **agree on each equivalence class** of $\equiv$.

The term $x\backslash x$ is denoted by 1_x.

Lemma 2

Let A be an rpo-semilattice and define $\equiv$ as above. Then each equivalence class of $\equiv$ **is a semilattice** $([x]_\equiv, \cdot)$ with identity element 1_x.

Note: In an ipo-semilattice $1_x = x\backslash x = -(x \cdot -x)$.

In an ipo-semilattice define $0_x = -1_x$ or equivalently $0_x = x \cdot -x$.

Lemma 3

Let A be an ipo-semilattice and define

$$\mathbb{B}_x = \{a \in A \mid 0_x \sqsubseteq a \sqsubseteq 1_x\}.$$

Then

1. the intervals $\mathbb{B}_x$ are closed under negation, i.e., $y \in \mathbb{B}_x \implies -y \in \mathbb{B}_x$,
2. $x \sqsubseteq y$ implies $0_x \sqsubseteq 0_y$ and $0_x \leq 0_y$,
3. $0_x \sqsubseteq 0_y$ if and only if $0_x \leq 0_y$,
4. $x \sqsubseteq y$ implies $1_x \sqsubseteq 1_y$, and
5. $1_x \cdot 1_y = 1_{xy}$.

ipo-semilattices are unions of Boolean algebras

Define $x + y = -(-y \cdot -x)$.

Theorem 1. Partition by Boolean Algebras

Given an ipo-semilattice A, the semilattice intervals $(\mathbb{B}_x, \cdot, +, -, 0_x, 1_x)$ are Boolean algebras and they **partition** A.

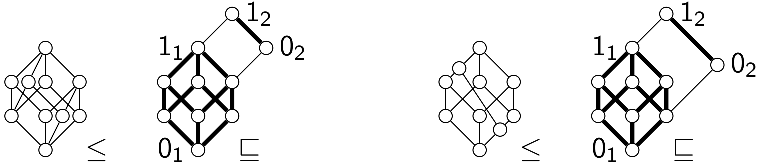
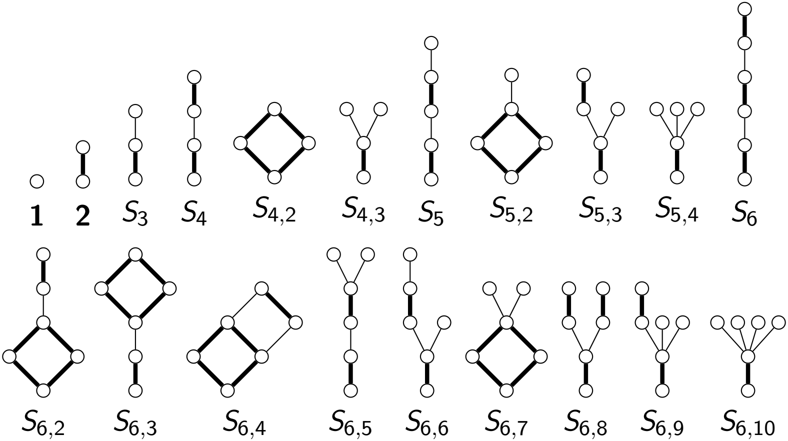


Figure: (Right) unital ipo-semilattice that is **not** an iℓ-semilattice.

The Boolean components are denoted by **thick lines** and are connected by homomorphisms (thin lines). For CldInRL the above theorem is due to [J., Tuyt, Valota 2021].

Note: A finite iℓ-semilattice is a (nonunital) **commutative idempotent involutive (i.e. Frobenius) quantale**.

Now we can construct all these algebras (only $\sqsubseteq$ is shown):



Subdirectly irreducible unital iℓ-semilattices

Lemma

Let $\mathbf{A}$ be a unital ipo-semilattice. If $0_x = 1_x$ then $x = 1$, hence all Boolean components except possibly the top one are nontrivial.

A unital ipo-semilattice is called **odd** if it satisfies the identity $-1 = 1$ (i.e., $0 = 1$).

Theorem 2.

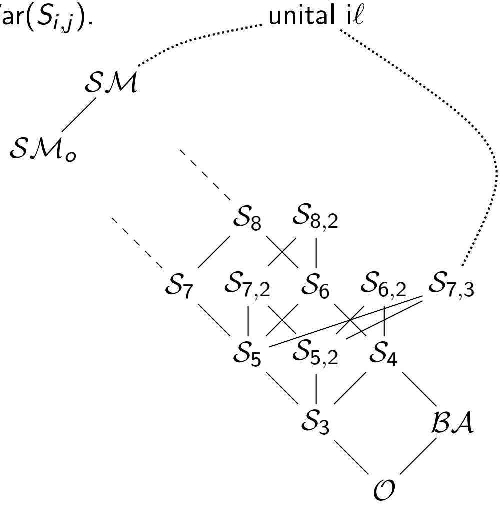
A finite unital ipo-semilattice $\mathbf{A}$ is odd if and only if $|A|$ is odd.

A finite unital iℓ-semilattice $\mathbf{A}$ is subdirectly irreducible if and only if 1 has a unique coatom in the monoidal order.

# of elem. $n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
unital iℓ-semilats	1	1	1	2	2	4	4	9	10	21	22	49	52	114	121	270
subdir. irreducible	0	1	1	1	2	2	4	4	9	10	21	22	49	52	114	121

Subvarieties of unital iℓ-semilattices

Let $\mathcal{S}_{i,j} = \mathsf{Var}(S_{i,j})$.



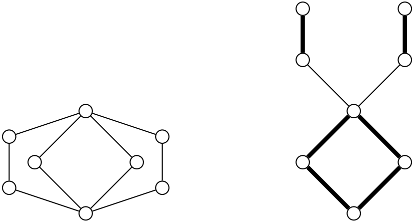
Theorem 3.

Let $\mathbf{A}$ be an iℓ-semilattice. Then for every $x \in A$ the multiplicative downset of 1_x is a **unital** sub-iℓ-semilattice.

Proof

- Let $\mathbf{A}_x$ denote the multiplicative downset of 1_x. If $y \cdot 1_x = y$ and $z \cdot 1_x = z$ then $(y \vee z) \cdot 1_x = (y \cdot 1_x) \vee (z \cdot 1_x) = y \vee z$ since $\cdot$ distributes over $\vee$. Therefore $\mathbf{A}_x$ is closed under join.
- Each Boolean component is closed under $-$, so it is clear that $\mathbf{A}_x$ is closed under $-$.
- By DeMorgan laws, closure under $-$ and $\vee$ guarantees closure under $\wedge$.

Therefore $\mathbf{A}_x$ is a sub-iℓ-semilattice.



$\leq$ $\sqsubseteq$

Figure: An 8-element iℓ-semilattice. Its multiplicative order shows its unital sub-iℓ-semilattices.

Semilattice direct systems and Płonka sums

A **semilattice direct system** (or **sd-system** for short) is a triple $\mathbf{B} = (\mathbf{B}_i, h_{ij}, I)$ such that

- I is a semilattice,
- $\{\mathbf{B}_i : i \in I\}$ is a family of **algebras of the same type** with **disjoint** universes,
- $h_{ij} : \mathbf{B}_i \to \mathbf{B}_j$ is a homomorphism for all $i \geq j \in I$ such that h_{ii} is the identity on $\mathbf{B}_i$ and for all $i \geq j \geq k$, $h_{jk} \circ h_{ij} = h_{ik}$.

The **Płonka sum** over $\mathbf{B}$ is the algebra $P_{ł}(\mathbf{B}) = \bigcup_{i \in I} \mathbf{B}_i$ with each fundamental operation $g^{\mathbf{B}}$ defined by

$$g^{\mathbf{B}}(b_{i_1}, \ldots, b_{i_n}) = g^{\mathbf{B}_j}(h_{i_1 j}(b_{i_1}), \ldots, h_{i_n j}(b_{i_n}))$$

where $b_{i_k} \in \mathbf{B}_{i_k}$ and $j = i_1 \cdots i_n$ is the semilattice meet of $i_1, \ldots, i_n \in I$.

iℓ-semilattices are multiplicative Płonka sums

Theorem 4.

Let $\mathbf{A}$ be an iℓ-semilattice, and define $I = (\{1_x \mid x \in A\}, \cdot)$. Then

1. $\mathbf{B} = (\mathbb{B}_i, h_{ij}, I)$ is a sd-system of Boolean algebras, where each $h_{ij} : \mathbb{B}_i \to \mathbb{B}_j$ is a generalized Boolean algebra homomorphism (i.e., mapping 1_i to 1_j but not 0_i to 0_j) defined by $h_{ij}(x) = x \cdot j$,
2. the image $h_{ij}[\mathbb{B}_i]$ is a proper filter,
3. the Płonka sum $P_{\l}(\mathbf{B})$ reconstructs the reduct algebra $(A, \cdot, -)$.

Reconstructing the lattice order takes more work.

Colimits of finite unital sub-iℓ-semilattices

Theorem 5.

Let $\mathbf{A}$ be a finite iℓ-semilattice, define $I = (\{1_x \mid x \in A\}, \cdot)$ and let A_i be the multiplicative downset of $i \in I$.
Then $\{A_i : i \in I\}$ is a system of finite unital subalgebras of A such that $A_i \cap A_j = A_{ij}$ and $A = \sum_{i \in I} A_i$.

By [J., Tuyt, Valota 2021] each finite unital iℓ-semilattice is determined by its monoidal semilattice, so the above theorem extends this result to nonunital iℓ-semilattices.

The same result is conjectured to hold for ipo-semilattices.

Dual Representation by Partial Functions Between Sets

Partial Functions

Definition. A **proper partial function** $f : X \to Y$ is a function from U to Y where $U \subsetneq X$ is the domain of f.

Developing a Dual Representation

Given an ipo-semilattice $\mathbf{A}$, it is a partition of Boolean components by Theorem 1.

Each Boolean component is determined by its set of atoms.

The partial functions map between sets of atoms (opposite to homomorphisms).

A dual representation of sd-systems of Boolean algebras gives a much more compact way of drawing finite ipo-semilattices.

Dual Representation by Partial Functions Between Sets

Every finite Boolean algebra $\mathbb{B}_i$ is **isomorphic** to the powerset Boolean algebra of its finite set X_i of atoms.

For $i \le j$, the **generalized BA homomorphism** h_{ji} corresponds to the **partial map** $f_{ij} : X_i \to X_j$ defined by

$$f_{ij}(a) = b \iff a \le h_{ji}(b) \text{ and } a \not\le h_{ji}(0_j).$$

A **sd-system of proper partial maps** is a triple $\mathbf{X} = (X_i, f_{ij}, I)$ such that

- I is a semilattice,
- $\{X_i : i \in I\}$ is a family of disjoint sets, and
- $f_{ij} : X_i \to X_j$ is a proper partial map for all $i \le j \in I$ such that $f_{ii} = id_{X_i}$ and for all $i \le j \le k$, $f_{jk} \circ f_{ij} = f_{ik}$.

Dual Representation by Partial Functions Between Sets

Lemma

In every ipo-semilattice $x, y \sqsubseteq z \implies 0_x \cdot 0_y = 0_{xy}$.

An sd-system of partial maps is **covering** if for all $i, j \leq k$ with $i \cdot j = \ell$, $\mathrm{dom}(f_{\ell,i}) \cup \mathrm{dom}(f_{\ell,j}) = X_\ell$.

Corollary

Every sd-system of partial maps of an ipo-semilattice is covering.

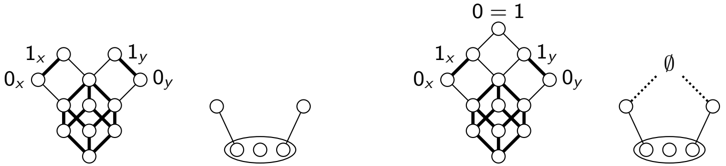


Figure: A nonunital ipo-semilattice that has no unital completion

Dual Representation by Partial Functions Between Sets

More Examples

The dual representations show clear differences even when the corresponding semilattice orders are visually similar.

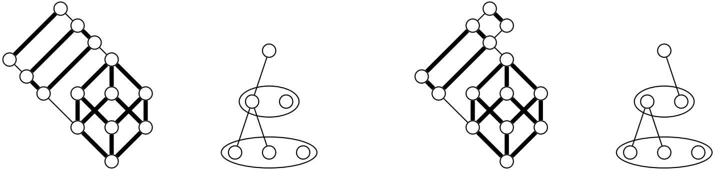


Figure: A pair of fourteen-element ipo-semilattices with identity and their dual representations.

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THANKS!