Filtral pretoposes and compact Hausdorff locales

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TACL 21(22), Coimbra, 20-24 June 2022

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Compact Hausdorff locales are significant for developing mathematics internally in a topos. Can we have a "pointless" characterization for the category of compact Hausdorff locales?

Hausdorff if its diagonal $X \to X \times X$ is closed. *Regular* if every $u \in OX$ equals the supremum of those $v \in OX$ that are well inside it, i.e such that $u \wedge v = 0$ and $\neg v \vee u = 1$.

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Universal sums: Joyal - Tierney, disjoint sums [KT].

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Closed quotients $f : Y \to X$ of subfit locales are subfit:

An open sublocale U of X corresponds to a nucleus $j = u \to -$ with inverse image $f^{-}j = f^{*}u \to -$ which is $f^{*}u \to - = \bigwedge_{i}(v_i \vee -)$ in the frame of nuclei on OY.

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If $w \leq f_{*}(f^{*}u \to f^{*}v)$, then $f^{*}w \leq f^{*}u \to f^{*}v$, equivalently $f^{*}(w \wedge u) \leq f^{*}v$, so we conclude that $w \leq u \to v$ by the surjectivity of f.

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Theorem

For a filtral pretopos $\mathcal{K}$, the functor $F \colon \mathcal{K} \to \mathrm{CHLoc}$ is full on subobjects, faithful, preserves (regular) epis and equalizers.

Preservation of equalizers (used also in showing that F is faithful): For a pair of maps $f, g : Y \to Z$ in $\mathcal{K}$ with equalizer $X \to Y$, the equalizer of $f[-]$, $g[-]$ is given as $\downarrow(\bigwedge\{f^{-1}[S] \vee g^{-1}[\sim S] \in OX \mid S \leq Z\})$ (Picado and Pultr, restated in terms of the genuine order of subobjects and taking into account that $\mathrm{Sub}(Z)$ is a co-Heyting algebra.)

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$$\begin{aligned} X \wedge (f^{-1}[S] \vee g^{-1}[\sim S]) &= (X \wedge f^{-1}[S]) \vee (X \wedge g^{-1}[\sim S]) \\ &= (X \wedge f^{-1}[S]) \vee (X \wedge f^{-1}[\sim S]) \\ &= X \wedge f^{-1}[S \vee \sim S]) = X \wedge Y = X \end{aligned}$$

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(Continued) Assume that the product $S = S_1 \times S_2$ of two filtral objects is filtral, the map $B_1 \coprod B_2 \to B$ involving the respective boolean algebras of complemented subobjects is injective. Then F preserves binary products. Assume further that the unique map to the terminal locale (which is compact Hausdorff) is a surjection, then F preserves the terminal object, hence all finite products.

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Obviously, the slice $\mathcal{K}/X$ does not have the property when $\mathcal{K}$ has it.

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But this can be seen in an elementary manner.

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Assuming CL, PIT and copowers of 1 in $\mathcal{K}$, the result of [MR] is recovered.

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What conditions on a g.m. give that the notion of filtral pretopos is stable under its inverse image?

ACKNOWLEDGEMENT

The speaker acknowledges the support of the General Secretariat of Research and Innovation (GSRI) and the Hellenic Foundation for Research and Innovation (HFRI).

This research is co-financed by Greece and the European Union (European Social Fund- ESF) through the Operational Programme "Human Resources Development, Education and Lifelong Learning" in the context of the project "Strengthening Human Resources Research Potential via Doctorate Research" (MIS-5000432), implemented by the State Scholarships Foundation (IKY).

THANK YOU.