Monotone-light factorizations and liftings

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Factorization systems

Let $\mathcal{C}$ be a category.

Definition (Factorization system)

Subclasses $\mathcal{L}, \mathcal{R}$ of $\mathrm{Mor}(\mathcal{C})$ such that any f can be uniquely factored as $f = r \circ l$, with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "l"] \arrow[ddr, bend right, "l'"'] & & Y \\
& Z \arrow[ur, "r"] \arrow[d, dashed, "\exists!"'] & \\
& Z' \arrow[uur, bend right, "r'"'] &
\end{tikzcd}$$

$\mathcal{L}, \mathcal{R}$ must also be closed under composition and contain all isos.

Examples

- (iso, all) and (all, iso) on all categories,
- (surjections, injections) on Set,
- (regular epi, mono) on regular categories,
- (inverted by F, cartesian) on $\mathcal{E}$ for a fibration $F\colon \mathcal{E} \to \mathcal{B}$

Monotone-light factorization in CHaus

Let $f\colon X \to Y$ be a continuous map.

Definition

- f is monotone if $f^{-1}(x)$ is connected for all x.
- f is light if $f^{-1}(x)$ is totally disconnected for all x.

Observe that

- Isomorphisms are monotone and light.
- Both properties are closed under composition.

Theorem [Eilenberg 31, Whyburn 50]

Every map f between compact Hausdorff spaces admits a unique factorization $r \circ l$ with r light and r monotone.

Monotone-light factorization in CHaus

$\mathcal{C}$ category w/ pullbacks, $p\colon E \to B$ morphism.

$$\begin{tikzcd}
p^*(X) \arrow[r] \arrow[d, "p^*(f)"'] & X \arrow[d, "f"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

p is of *effective descent* iff $p^*\colon \mathcal{C} \downarrow B \to \mathcal{C} \downarrow E$ is monadic.

Consider the fact. sys. $(\mathcal{L}, \mathcal{R})$ given by $\pi_0\colon \mathrm{CHaus} \to \mathrm{Stn}$.

Theorem (Carboni, Janelidze, Kelly, Paré, 97)

Let $f\colon X \to B$ be a continuous map of compact Hausdorff spaces.

- *f is monotone iff for all p, $p^*(f) \in \mathcal{L}$*
- *f is light iff there exists p eff.desc. s.t. $p^*(f) \in \mathcal{R}$.*

Monotone-light factorizations in AbGrp

Let A be an abelian group.

Recall $\operatorname{ord} g = \#\langle g \rangle$. A is a *torsion* group if for all $g \in A$, $\operatorname{ord} g$ is finite, and A is *torsion-free* if the only torsion element is the unit.

Replacing totally disconnected $\to$ torsion-free, connected $\to$ torsion, we get an analogous result and proof for a ML factorization system in AbGrp.

Stabilization and localization

Let $\mathcal{P}$ be a property of maps of $\mathcal{C}$ with pullbacks.

$$\begin{tikzcd}
p^*(X) \arrow[r] \arrow[d, "p^*(f)"'] & X \arrow[d, "f"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

Stabilization: $f \in \mathcal{P}_{\mathrm{stab}}$ iff $p^*(f) \in \mathcal{P}$ for all p.

Localization: $f \in \mathcal{P}_{\mathrm{loc}}$ iff $p^*(f) \in \mathcal{P}$ for some p of effective descent.

Monotone-light factorization systems

Let $(\mathcal{L}, \mathcal{R})$ be a factorization system on $\mathcal{C}$ with pullbacks.

Definition (Monotone-light)

The *monotone-light factorization system associated* to $(\mathcal{L}, \mathcal{R})$ is $(\mathcal{L}_{\mathsf{stab}}, \mathcal{R}_{\mathsf{loc}})$ whenever this is a factorization system.

When is $(\mathcal{L}_{\mathsf{stab}}, \mathcal{R}_{\mathsf{loc}})$ a factorization system?

Monotone-light factorization system

Problems:

- $\mathcal{R}_{\mathrm{loc}}$ might not be closed under composition!
- Factorization cannot always be guaranteed!

Positive results:

- If $\mathcal{L} = \mathcal{L}_{\mathrm{stab}}$, then $\mathcal{R}_{\mathrm{loc}} = \mathcal{R}$.
- $(\mathcal{L}_{\mathrm{stab}}, \mathcal{R}_{\mathrm{loc}})$ is a fact. sys. iff every morphism has a locally stable $(\mathcal{L}, \mathcal{R})$-factorization.

An example of a lifting

Consider the fibration $\pi \colon \mathsf{Cat} \to \mathsf{Ord}$

Theorem (Xarez, 03)

The induced fact. sys. $(\mathcal{L}, \mathcal{R})$ has an associated ML fact. sys..

Likewise, consider $\pi_! \colon \mathsf{Cat}\text{-}\mathsf{Cat} \to \mathsf{Ord}\text{-}\mathsf{Cat}$.

Theorem (Xarez, 22)

The induced fact. sys. $(\mathcal{L}_!, \mathcal{R}_!)$ has an associated ML fact. sys..

Moreover, we can observe that $((\mathcal{L}_{\mathsf{stab}})_!, (\mathcal{R}_{\mathsf{loc}})_!) = ((\mathcal{L}_!)_{\mathsf{stab}}, (\mathcal{R}_!)_{\mathsf{loc}})$.

Lemma

Let $\mathcal{V}$ be a monoidal category, and let $(\mathcal{L}, \mathcal{R})$ be a fact. sys. on the underlying category, and consider the following properties of $\mathcal{V}$-functors:

- $\mathcal{L}_! = \{\, F \mid F \text{ bijective on objects and } F_{x,y} \in \mathcal{L} \,\}$,
- $\mathcal{R}_! = \{\, F \mid F_{x,y} \in \mathcal{R} \,\}$.

*Then, for suitable $\mathcal{L}, \mathcal{R}$, this defines a fact. sys. on $\mathcal{V}$-*Cat.

Consequently, if $\mathcal{L}$ is stable then so is $\mathcal{L}_!$.

Lifting properties of reflections

Semi-left exact reflection: $L \dashv R$ with R fully faithful, L fibration.
Reflection w/ stable units: $L \dashv R$ with R fully faithful, $\eta_x \in \mathcal{L}_{\mathsf{stab}}$ for all x.

Lemma

Consider $\mathcal{V} \mapsto \mathcal{V}$-Cat*.*

This 2-functor maps semi-left exact reflections and reflections with stable unit (in a suitable sense) to the respective classical notion.

Moreover, if $L \dashv R$ *is a suitable semi-left exact reflection, then the induced* $(\mathcal{L}, \mathcal{R})$ *is also suitable, and* $(\mathcal{L}_!, \mathcal{R}_!)$ *is the fact. sys. induced by* $L_!$*.*

Probabilistic metric spaces

The quantale of distribution functions is given by

$$\Delta = \left\{ f \colon [0,\infty] \to [0,1] \;\middle|\; fx = \bigvee_{y<x} fy \right\},$$

with pointwise order and operation.

We have a (monoidal) fibration $\Delta \to [0,\infty]$, inducing a stable factorization system.

Open problem

Assume $(\mathcal{L}, \mathcal{R})$ admits ML fact. sys.:

$$\begin{tikzcd}
(\mathcal{L}, \mathcal{R}) \arrow[r, mapsto, "(-)\text{-Cat}"] \arrow[d, "\text{stab,loc}"] & (\mathcal{L}_!, \mathcal{R}_!) \arrow[d, dashed, "?"] \\
(\mathcal{L}_{\text{stab}}, \mathcal{R}_{\text{loc}}) \arrow[r, mapsto, "(-)\text{-Cat}"] & ((\mathcal{L}_{\text{stab}})_!, (\mathcal{R}_{\text{loc}})_!)
\end{tikzcd}$$

Even though $(\mathcal{L}_{\text{stab}})_! = (\mathcal{L}_!)_{\text{stab}}$, only $(\mathcal{R}_!)_{\text{loc}} \subseteq (\mathcal{R}_{\text{loc}})_!$ is guaranteed.

Question: when do we have the reverse inclusion?

Future work

Let $\mathbb{B}$ be a 2-category with the comma objects $\mathrm{id}_c \downarrow f$ for all $f\colon b \to c$.

This is enough to internalize fibrations and factorization systems.

A pseudofunctor $F\colon \mathbb{B} \to \mathbb{C}$ preserving the comma object will also preserve those notions.

MonCat and $\mathrm{MonCat}_{\mathrm{lax}}$ both have these bilimits.

$(-)\text{-Cat}\colon \mathrm{MonCat}_{\mathrm{lax}} \to \mathrm{CAT}$ does not preserve them, but $\mathrm{MonCat}_{\mathrm{lax}} \to \mathrm{CAT} \downarrow \mathrm{Set}$ does.

Thank you!