Some Topological Considerations on Orthogonality

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Orthogonal relations

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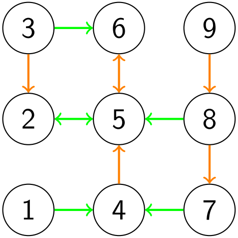
We call these (indistinctly) orthogonal frames.

A simple example

Perhaps this is better illustrated with a drawing:



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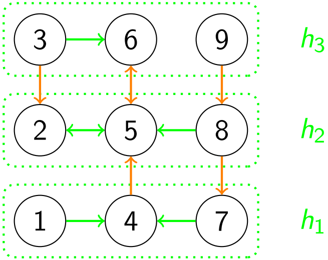

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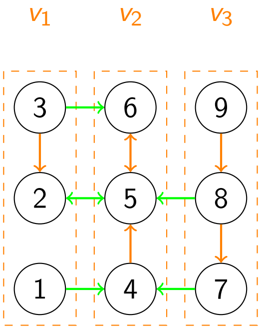

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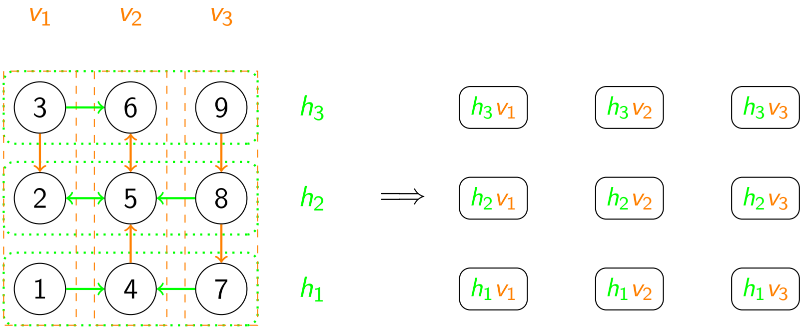

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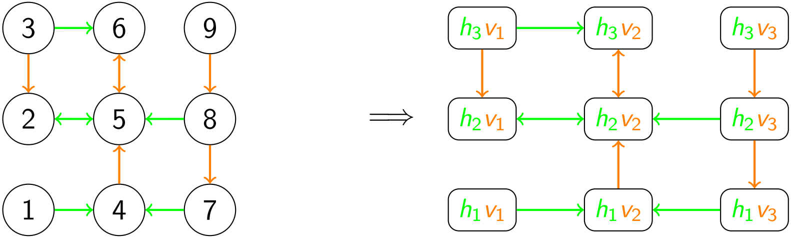

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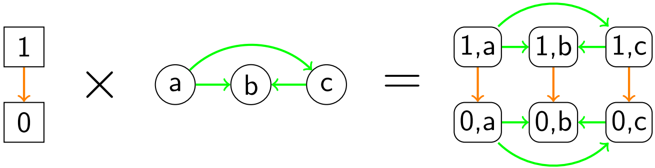

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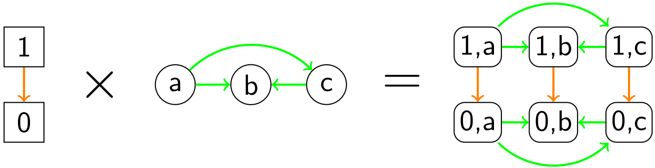

- Models for STIT logics (Belnap et al., 2001) and Social Epistemic Logics (Seligman et al., 2011) are generally orthogonal.

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Definition: Subset Space

A tuple (X, σ) with $\varnothing \neq \sigma \subseteq \mathcal{P}(X)$.
σ is a topology if it is closed under unions, finite intersections, and contains $\varnothing$ and X.

Some basic notions of Subset Space Logic

Given the following:

- a subset space (X, σ),
- a language including two modalities K and $\Box$, and
- $U \in \sigma$ and $x \in U$,

the semantics of SSL (Moss & Parikh, 1992) read as follows:

Semantics of SSL

$x, U \models K\phi$ iff $(y \in U \Rightarrow y, U \models \phi)$;
$x, U \models \Box\phi$ iff $(x \in V \in \sigma \,\&\, V \subseteq U \Rightarrow x, V \models \phi)$.

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Now, get a subset space and define two relations on the set

$$\{(x, U) : x \in U \;\&\; U \in \sigma\} :$$

- $(x, U) \geq (y, V)$ iff $x = y$ and $U \supseteq V$;
- $(x, U) \sim (y, V)$ iff $U = V$.

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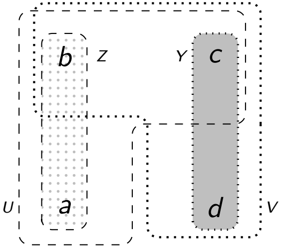
... and moreover $\sim$ and $\geq$ are orthogonal relations!

Perhaps clearer with a drawing



$$\begin{array}{ccc}
c, U & & \\
\sim_\sigma & & \\
b, U & \geq_X & b, Z \\
\sim_\sigma & & \sim_\sigma \\
a, U & \geq_X & a, Z
\end{array}
\qquad
\begin{array}{ccc}
d, V & \geq_X & d, Y \\
\sim_\sigma & & \sim_\sigma \\
c, V & \geq_X & c, Y \\
\sim_\sigma & & \\
b, V & &
\end{array}$$

$$U = \{a, b, c\}, V = \{b, c, d\}, Z = \{a, b\}, Y = \{c, d\}$$

First question

This brings us to the first question:

What are some necessary and sufficient conditions on some orthogonal frame (X, R_1, R_2) for it to be isomorphic to the relational structure generated by some subset/topological space?

Or, in categorical terms:

What is the class of orthogonal frames which is categorically equivalent to subset/topological spaces?

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(1) $\equiv \cap \sim = Id_{\mathcal{O}}$;

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(4) for all nonempty $A \subseteq \mathcal{O}$, closed under $\sim$, there is some b such that
(4.1) $\forall a \in A$: $a(\equiv \circ \sim)b$;

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(4) for all nonempty $A \subseteq \mathcal{O}$, closed under $\sim$, there is some b such that *(4.1)* $\forall a \in A$: $a(\equiv \circ \sim)b$; *(4.2)* $\forall b' \sim b\ \exists a' \in A$: $a' \equiv b'$.

Preorder from the equivalence relation

Remark

Note that, a few slides ago, I gave the relational structure associated to a subset/topological space relative to an equivalence relation $\sim$ and a *preoder* $\leq$.
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This preorder can be defined from the two equivalence relations given in the last slide:

$$a \leq b \text{ iff } a \equiv b \text{ and } a'(\equiv \circ \sim)b \text{ for all } a' \sim a.$$

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In the particular case of topological spaces, we can take the preorder $\leq$ as a primitive and simply define $\equiv$ as $\leq \circ \geq$.
... but let's forget about this for now.

The results

Theorem

An orthogonal subset (resp. topological) frame is isomorphic to the relational structure associated to some subset (resp. topological) space; conversely, every such structure is itself an orthogonal subset/topological frame.

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Proof sketch

The corresponding space is $(X_{\mathcal{O}}, \sigma_O)$, where
$X_{\mathcal{O}}$ is the quotient set $\mathcal{O}/_{\equiv}$, and
$\sigma_\pi = \{\varnothing\} \cup \{U_\pi : \pi \in \mathcal{O}/_{\sim}\}$, where

$$U_\pi := \{\sigma \in X_{\mathcal{O}} : \sigma \cap \pi \neq \varnothing\}.$$

We note that, by orthogonality, $\theta \in U_\pi$ if and only if $\theta \cap \pi$ is a singleton, which provides us a natural way to construct the isomorphism.

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For the 0 people in this room who know less point-free topology than me:

Definition

In the context of point-free topology, a frame is a complete lattice $(L, \leq)$ such that, for all $A \subseteq L$, $b \in L$: $(\bigvee A) \wedge b = \bigvee_{a \in A}(a \wedge b)$.
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1. Every equivalence class $[x]_{\equiv}$ constitutes a botomless frame with respect to the preorder $\leq$ defined in the previous slide;
2. The quotient set $\mathcal{O}/_{\sim}$ constitutes a botomless frame along with the relation $[a]_{\sim} \preceq [b]_{\sim}$ iff $a(\leq \circ \sim)b$; moreover, this botomless frame is isomorphic to the lattice $(\sigma \setminus \{\varnothing\}, \subseteq)$, induced by the original topology.

Third observation

With this correpondence in mind, it now makes sense to define some topological notions on certain classes of birelational structures! Just to give a few examples:

- an orthogonal frame $(\mathcal{O}, \equiv, \sim)$ satisfies the T_2 separation axiom if, for all $a, b \in \mathcal{O}$ such that $a \not\equiv b$, there exist some $c \equiv a$ and $d \equiv b$ such that, for all $c' \sim c$ and $d' \sim d$, $c' \not\equiv d'$.
- an orthogonal frame is Alexandroff if, for every nonempty set $A \subseteq \mathcal{O}$, there exists some $b \in \mathcal{O}$ such that $b(\leq \circ \sim)a$ for all $a \in A$.

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This is not a rhetorical question, **please help**.

I do have some (very raw) ideas

The correspondence between the three definitions of 'orthogonal frames' has allowed us to prove some new stuff or to simplify some existing proofs:

- Standard canonical model completeness proof of Social Epistemic Logic (Balbiani & Fernández González, 2021)
- Showing that the logic of certain 'gimmicky' models for STIT logics is, thanks to this correspondence, an already-known logic (Fernández González & Lorini, forthcoming)

And on the topological side of things?

Many logics for knowledge, belief, evidence whose models are based on topological spaces have completeness proofs that tend to be rather involved and nonstandard.

Is it possible that the observations made here will allow us, similarly to the previous examples, to apply standard, round-of-the-mill techniques to rather esoteric classes of models?

I hope so! But, again... I need help!

Obrigado!