Internal Factorisation Systems

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Overview

Internal Categories

Let $\mathbb{C}$ be an category with pullbacks. An **internal category**, C, in $\mathbb{C}$ is a diagram

$$\begin{tikzcd} C_0 \arrow[r, "e"] & C_1 \arrow[l, bend right, "d"'] \arrow[l, bend left, "c"] & C^{\leftarrow\leftarrow} \arrow[l, "m"'] \end{tikzcd}$$

C_0: Object of objects
C_1: Object of morphisms
$C^{\leftarrow\leftarrow}$: Object of composable morphisms
d: Domain morphism
c: Codomain morphism
e: Morphism of identities
m: Composition morphism

$$\begin{tikzcd} C^{\leftarrow\leftarrow} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "c"] \\ C_1 \arrow[r, "d"] & C_0 \end{tikzcd}$$

Internal Categories

Such that the morphisms satisfy the following four commutative diagrams

$$\begin{tikzcd}
C_0 \arrow[r, "e"] \arrow[d, "e"] \arrow[dr, "1_{C_0}"] & C_1 \arrow[d, "d"] \\
C_1 \arrow[r, "c"] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[d, "c"] & C^{\leftarrow\leftarrow} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "m"] & C_1 \arrow[d, "d"'] \\
C_0 & C_1 \arrow[l, "c"'] \arrow[r, "d"] & C_0
\end{tikzcd}$$

$$\begin{tikzcd}
C_1 \arrow[r, "\langle ec, 1_{C_1} \rangle"] \arrow[d, "\langle 1_{C_1}, ed \rangle"'] \arrow[dr, "1_{C_1}"] & C^{\leftarrow\leftarrow} \arrow[d, "m"] \\
C_1 \times_{C_0} C_1 \arrow[r, "m"] & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C^{\leftarrow\leftarrow\leftarrow} \arrow[r, "m \times 1_{C_1}"] \arrow[d, "1_{C_1} \times m"] & C^{\leftarrow\leftarrow} \arrow[d, "m"] \\
C^{\leftarrow\leftarrow} \arrow[r, "m"] & C_1
\end{tikzcd}$$

where $C^{\leftarrow\leftarrow\leftarrow}$ is defined as the pullback

$$\begin{tikzcd}
C^{\leftarrow\leftarrow\leftarrow} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\leftarrow\leftarrow} \arrow[d, "\pi_1"'] \\
C^{\leftarrow\leftarrow} \arrow[r, "\pi_2"] & C_0
\end{tikzcd}$$

Orthogonality

Let $\mathbb{C}$ be a category, and let $f : X \to Y$ and $g : X' \to Y'$ be two morphisms in $\mathbb{C}$. f is **orthogonal** to g, written $f \downarrow g$, if for all morphisms $u : X \to X'$ and $v : Y \to Y'$ in $\mathbb{C}$ with $vf = gu$, there exists a unique morphism $z : Y \to X'$ such that $u = zf$ and $v = gz$, as in the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \arrow[dl, dashed, "z"'] \\ X' \arrow[r, "g"] & Y' \end{tikzcd}$$

Orthogonality

For two morphisms f and g in a category $\mathbb{C}$, with $f \downarrow g$, we have a correspondence between the diagrams:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[dl] \\ X' \arrow[r, "g"] & Y' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "g"] & Y' \end{tikzcd}$$

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[**Kelly**] For two morphisms $f : X \to X'$ and $g : Y \to Y'$ in a category $\mathbb{C}$, we have that $f \downarrow g$ if and only if the following square is a pullback in **Sets**:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{C}}(Y, X') \arrow[r, "\mathrm{Hom}_{\mathbb{C}}(f{,}X')"] \arrow[d, "\mathrm{Hom}_{\mathbb{C}}(Y{,}g)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Hom}_{\mathbb{C}}(X, X') \arrow[d, "\mathrm{Hom}_{\mathbb{C}}(X{,}g)"] \\ \mathrm{Hom}_{\mathbb{C}}(Y, Y') \arrow[r, "\mathrm{Hom}_{\mathbb{C}}(f{,}Y')"'] & \mathrm{Hom}_{\mathbb{C}}(X, Y') \end{tikzcd}$$

Orthogonality

Let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms of a category $\mathbb{C}$. Then $\mathcal{E}$ is **orthogonal** $\mathcal{M}$, written $\mathcal{E} \downarrow \mathcal{M}$, if for all morphisms e in $\mathcal{E}$ and m in $\mathcal{M}$, we have that $e \downarrow m$.

Factorisations

Let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms of a category $\mathbb{C}$.

An $(\mathcal{E}, \mathcal{M})$-**factorisation** of a morphism $f : A \to B$ in $\mathbb{C}$ is a pair of morphisms $e : A \to I$ in $\mathcal{E}$ and $m : I \to B$ in $\mathcal{M}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\ & I \arrow[ur, "m"'] & \end{tikzcd}$$

We say that $\mathbb{C}$ **has** $(\mathcal{E}, \mathcal{M})$-**factorisations** if every morphism of $\mathbb{C}$ has an $(\mathcal{E}, \mathcal{M})$-factorisation.

Factorisation System

Let $\mathbb{C}$ be a category and let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms of $\mathbb{C}$. Then the pair $(\mathcal{E}, \mathcal{M})$ forms a **factorisation system** on $\mathbb{C}$ if the following four conditions are met:

FS1. $\mathcal{E}$ and $\mathcal{M}$ contain all the isomorphisms of $\mathbb{C}$.

FS2. $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.

FS3. $\mathcal{E} \downarrow \mathcal{M}$.

FS4. $\mathbb{C}$ has $(\mathcal{E}, \mathcal{M})$-factorisations.

Subobjects of morphisms

Let $\mathbb{C}$ be a category with pullbacks. Consider an internal category C in $\mathbb{C}$:

$$\begin{tikzcd} C_0 \arrow[r, "e" description] & C_1 \arrow[l, bend right, "d"'] \arrow[l, bend left, "c"] & C^{\leftarrow\leftarrow} \arrow[l, "m"'] \end{tikzcd}$$

A **subobject of morphisms** of C is a subobject of C_1 in $\mathbb{C}$.

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We therefore consider two subobjects of C_1:

$$\varepsilon : E \to C_1 \quad \text{and} \quad \mu : M \to C_1$$

Subobjects of morphisms

The **subobject of all morphisms** of an internal category C is the subobject $1_{C_1} : C_1 \to C_1$.

The **subobject of identity morphisms** of an internal category C is the subobject $e : C_0 \to C_1$.

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For two subobjects of morphisms $\alpha : A \to C_1$ and $\beta : B \to C_1$ of an internal category C, we say that α **contains** β if $\beta \leq \alpha$ as subobjects of C_1:

$$\begin{tikzcd} A \arrow[r, "\alpha"] & C_1 \\ B \arrow[u, dashed, "\beta_\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

Subobjects of morphisms

The **object of composable morphisms** for two subobject of morphisms $\alpha : A \to C_1$ and $\beta : B \to C_1$ is

$$\begin{tikzcd} B^{\leftarrow} A^{\leftarrow} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "c\alpha"] \\ B \arrow[r, "d\beta"] & C_0 \end{tikzcd}$$

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In the case that $\alpha = \beta$, we will write $A^{\leftarrow\leftarrow}$ for $A^{\leftarrow}A^{\leftarrow}$.

We similarly define $D^{\leftarrow}B^{\leftarrow}A^{\leftarrow}$ for subobjects of morphisms $\alpha : A \to C_1$, $\beta : B \to C_1$ and $\delta : D \to C_1$.

Subobject of Isomorphisms

Let C be an internal category in a category $\mathbb{C}$ with pullbacks.

The **object of points** of C is the pullback:

$$\begin{tikzcd} \mathrm{Pt}(C) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\leftarrow\leftarrow} \arrow[d, "m"] \\ C_0 \arrow[r, "e"] & C_1 \end{tikzcd}$$

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C_0 \arrow[r, "e"] & C_1
\end{tikzcd}
$$

The **object of isomorphisms** of C is the pullback:

$$
\begin{tikzcd}
\mathrm{Iso}(C) \arrow[rr, "\pi_2"] \arrow[dd, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{Pt}(C) \arrow[d, "\pi_2"] \\
& & C^{\leftarrow\leftarrow} \arrow[d, "\pi_1"] \\
\mathrm{Pt}(C) \arrow[r, "\pi_2"] & C^{\leftarrow\leftarrow} \arrow[r, "\pi_2"] & C_1
\end{tikzcd}
$$

Subobject of Isomorphisms

We define σ as the compositions:

$$\sigma: \ \mathrm{Iso}(C) \xrightarrow{\pi_1} \mathrm{Spl}(C) \xrightarrow{\pi_2} C^{\leftarrow\leftarrow} \xrightarrow{\pi_1} C_1$$

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Proposition: σ is a monomorphism.

We refer to $\sigma : \mathrm{Iso}(C) \to C_1$ as the **subobject of isomorphisms**.

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Proposition: σ is a monomorphism.

We refer to $\sigma : \mathrm{Iso}(C) \to C_1$ as the **subobject of isomorphisms**.

For a subobject of morphisms $\alpha : A \to C_1$ of an internal category C, we say that α **contains all isomorphisms** of C if α contains σ.

Closure under composition

A subobject of morphisms $\alpha : A \to C_1$ of an internal category C is **closed under composition** if there exists a morphism $m_\alpha : A^{\leftarrow\leftarrow} \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
A^{\leftarrow\leftarrow} \arrow[r, dashed, "m_\alpha"] \arrow[d, "\alpha\times\alpha"'] & A \arrow[d, "\alpha"] \\
C^{\leftarrow\leftarrow} \arrow[r, "m"] & C_1
\end{tikzcd}$$

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A^{\leftarrow\leftarrow} \arrow[r, dashed, "m_\alpha"] \arrow[d, "\alpha\times\alpha"'] & A \arrow[d, "\alpha"] \\
C^{\leftarrow\leftarrow} \arrow[r, "m"] & C_1
\end{tikzcd}
$$

This composition morphism, m_α, inherits the associativity of m. That is, the following diagram commutes:

$$
\begin{tikzcd}
A^{\leftarrow\leftarrow\leftarrow} \arrow[r, "1\times m_\alpha"] \arrow[d, "m_\alpha\times 1"'] & A^{\leftarrow\leftarrow} \arrow[d, "m_\alpha"] \\
A^{\leftarrow\leftarrow} \arrow[r, "m_\alpha"] & A
\end{tikzcd}
$$

Closure under composition

Proposition: The subobject of all morphisms, 1_{C_1}, the subobject of identity morphisms, e, and the subobject of isomorphisms, σ, are all closed under composition.

Orthogonality

Let C be an internal category and let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. Then ε is **orthogonal** to μ, written $\varepsilon \downarrow \mu$ if the following diagram is a pullback

$$\begin{tikzcd}
M^{\leftarrow} C_1^{\leftarrow} E^{\leftarrow} \arrow[r, "m(\mu\times 1)\times 1"] \arrow[d, "1\times m(1\times\varepsilon)"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1^{\leftarrow} E^{\leftarrow} \arrow[d, "m(1\times\varepsilon)"] \\
M^{\leftarrow} C_1^{\leftarrow} \arrow[r, "m(\mu\times 1)"] & C_1
\end{tikzcd}$$

Factorisation

Let C be an internal category and let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. Then C **has** (ε, μ)**-factorisations** if there exists a morphism $\tau : C_1 \to M^{\leftarrow}E^{\leftarrow}$ such that $m(\mu \times \varepsilon)\tau = 1_{C_1}$.

This is equivalent to requiring $m(\mu \times \varepsilon)$ to be a split epimorphism, with a specified splitting.

In **Sets**, due to the Axiom of Choice, one only requires $m(\mu \times \varepsilon)$ be a epimorphism, so why not require only this in general?

Internal Factorisation System

Let C be an internal category in a category $\mathbb{C}$ with pullbacks and let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. The pair (ε, μ) forms an **internal factorisation system** on C if:

IFS1. ε and μ contain all isomorphisms of C: There exist morphisms σ_ε and σ_μ such the following triangles commute

$$\begin{tikzcd} E \arrow[r, "\varepsilon"] & C_1 \\ \mathrm{Iso}(C) \arrow[u, dashed, "\sigma_\varepsilon"] \arrow[ur, "\sigma"'] & \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\mu"] & C_1 \\ \mathrm{Iso}(C) \arrow[u, dashed, "\sigma_\mu"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

IFS2. ε and μ are closed under composition: There exist morphism m_ε and m_μ such that the following squares commute:

$$\begin{tikzcd} E^{\leftarrow\leftarrow} \arrow[r, dashed, "m_\varepsilon"] \arrow[d, "\varepsilon\times\varepsilon"'] & E \arrow[d, "\varepsilon"'] \\ C^{\leftarrow\leftarrow} \arrow[r, "m"] & C_1 \end{tikzcd} \qquad \begin{tikzcd} M^{\leftarrow\leftarrow} \arrow[r, dashed, "m_\mu"] \arrow[d, "\mu\times\mu"'] & M \arrow[d, "\mu"'] \\ C^{\leftarrow\leftarrow} \arrow[r, "m"] & C_1 \end{tikzcd}$$

Internal Factorisation System

IFS3. $\varepsilon \downarrow \mu$: The following square is a pullback:

$$\begin{tikzcd} M^{\leftarrow} C_1^{\leftarrow} E^{\leftarrow} \arrow[r, "m(\mu\times 1)\times 1"] \arrow[d, "1\times m(1\times\varepsilon)"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1^{\leftarrow} E^{\leftarrow} \arrow[d, "m(1\times\varepsilon)"] \\ M^{\leftarrow} C_1^{\leftarrow} \arrow[r, "m(\mu\times 1)"] & C_1 \end{tikzcd}$$

IFS4. C has (ε, μ)-factorisations: There exists a morphism τ such that $m(\mu \times \varepsilon)\tau = 1_{C_1}$

The trivial internal factorisation system

If C is an internal category in a finitely complete category $\mathbb{C}$, then the pair $(\sigma, 1_{C_1})$ forms an internal factorisation system on C.

The Intersection of ε and μ

For a usual factorisation system $(\mathcal{E}, \mathcal{M})$, the intersection of the two classes, $\mathcal{E} \cap \mathcal{M}$ is the class of isomorphisms.

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Internally, we have that:

Proposition: If (ε, μ) is an internal factorisation system on an internal category C, the following square is a pullback:

$$\begin{tikzcd} \mathrm{Iso}(C) \arrow[r, "\sigma_\mu"] \arrow[d, "\sigma_\varepsilon"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "\mu"] \\ E \arrow[r, "\varepsilon"] & C_1 \end{tikzcd}$$

Essential Uniqueness of Factorisations

For a usual factorisation system $(\mathcal{E}, \mathcal{M})$, $(\mathcal{E}, \mathcal{M})$-factorisations are unique up to isomorphism. That is, if $f = me = m'e'$ are two factorisations of f, then there exists an isomorphism φ making the following diagram commute:

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e"] \arrow[ddr, bend right, "e'"'] & & B \\
& I \arrow[ur, "m"'] \arrow[d, dashed, "\varphi"'] & \\
& I' \arrow[uur, bend right, "m'"'] &
\end{tikzcd}
$$

Essential Uniqueness of Factorisations

Internally:

Proposition: If (ε, μ) is an internal factorisation system on an internal category C, then the following diagram is a pullback:

$$
\begin{tikzcd}
M^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow}E^{\leftarrow} \arrow[r, "1\times m_\varepsilon(\sigma_\varepsilon\times 1)"] \arrow[d, "m_\mu(1\times\sigma_\mu)\times 1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M^{\leftarrow}E^{\leftarrow} \arrow[d, "m(\mu\times\varepsilon)"] \\
M^{\leftarrow}E^{\leftarrow} \arrow[r, "m(\mu\times\varepsilon)"'] & C_1
\end{tikzcd}
$$

Note that:

We only require $\mathrm{IFS1}$ and $\mathrm{IFS2}$ to define this notion.
We only require $\mathrm{IFS1}$, $\mathrm{IFS2}$ and $\mathrm{IFS3}$ to prove this proposition.

Essential Uniqueness of Factorisations

Let C be an internal category in a category $\mathbb{C}$ with pullbacks. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. TFAE:

1. (ε, μ) forms an internal factorisation system on C.
2. (ε, μ) satisfies $\mathrm{IFS1}$, $\mathrm{IFS2}$, $\mathrm{IFS4}$ and
   $\mathrm{IFS3}^*$: (ε, μ)-factorisations are unique up to isomorphism.

The Cancellation Properties

For a usual factorisation system $(\mathcal{E}, \mathcal{M})$, $\mathcal{E}$ satisfies the **right cancellation property**:

if gf and f are in $\mathcal{E}$, then g is in $\mathcal{E}$,

and $\mathcal{M}$ satisfies the **left cancellation property**:

if gf and g are in $\mathcal{M}$, then f is in $\mathcal{M}$.

The cancellation properties

We may define internal versions of these properties, and for an internal factorisation system (ε, μ) on an internal category C, ε and μ respectively satisfy them:

Proposition: The following squares are pullbacks:

$$
\begin{tikzcd}
E^{\leftarrow\leftarrow} \arrow[r, "m_\varepsilon"] \arrow[d, "\varepsilon\times 1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\varepsilon"] \\
C_1^{\leftarrow} E^{\leftarrow} \arrow[r, "m(1\times\varepsilon)"] & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
M^{\leftarrow\leftarrow} \arrow[r, "m_\mu"] \arrow[d, "1\times\mu"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "\mu"] \\
M^{\leftarrow} C_1^{\leftarrow} \arrow[r, "m(\mu\times 1)"] & C_1
\end{tikzcd}
$$

ε and μ determine each other

Let (ε, μ) be an internal factorisation system on an internal category C and let $\varepsilon' : E' \to C_1$ and $\mu' : M' \to C_1$ be two subobjects of morphisms of C. Then:

$\varepsilon \downarrow \mu'$ if and only if $\mu' \leq \mu$.

$\varepsilon' \downarrow \mu$ if and only if $\varepsilon' \leq \varepsilon$.

ε and μ determine each other

If (ε, μ) and (ε', μ') are two internal factorisation systems on an internal category C, then:

$$\varepsilon' \leq \varepsilon \text{ if and only if } \mu \leq \mu'.$$

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We may thus define an order on the internal factorisation systems on an internal category C by:

$$(\varepsilon, \mu) \leq (\varepsilon', \mu') \text{ iff } \mu \leq \mu'.$$

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If (ε, μ) and (ε', μ') are two internal factorisation systems on an internal category C, then:

$$\varepsilon' \leq \varepsilon \text{ if and only if } \mu \leq \mu'.$$

We may thus define an order on the internal factorisation systems on an internal category C by:

$$(\varepsilon, \mu) \leq (\varepsilon', \mu') \text{ iff } \mu \leq \mu'.$$

Moreover, we have that:

$$\varepsilon \sim \varepsilon' \text{ if and only if } \mu \sim \mu'.$$

Mal'tsev Categories

A **Mal'tsev category** is a category for which every internal reflexive relation is an internal equivalence relation.

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Every internal factorisation system on an internal groupoid is trivial, $(\sigma, 1_{C_1})$.

[**Brown-Spencer**]: $\mathrm{Cat}(\mathbf{Grp}) \sim \mathbf{XMod}$, so we do not obtain factorisation systems for crossed modules

Schreier Internal Categories and Crossed Semimodules

A **Schreier internal category** in **Mon** is an internal category C which satisfies:

$$(\forall\, f \in C_1)(\exists!\, k \in \mathrm{Ker}(d))\; f = k + ed(f)$$

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A **crossed semimodule** is a quadruple (A, B, α, f) where:

- A and B are monoids
- α is a (left) monoid action of B on A
- $f : A \to B$ is a monoid homomorphisms
- Satisfying, for all $a, a' \in A$ and $b \in B$:
  1. $f({}^{b}a) + b = b + f(a)$ (Equivariance)
  2. ${}^{f(a)}a' + a = a + a'$ (Peiffer Identity)

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[Patchkoria]: $\mathrm{SCat}(\textbf{Mon}) \sim \textbf{XSMod}$

Schreier Internal Factorisation Systems

A subobject of morphisms $\alpha : A \to C_1$ of a Schreier internal category C in **Mon** is **Schreier** if for all $a \in A$, with $a = k_a + ed(a)$ for $k_a \in \mathrm{Ker}(d)$, we have that $k_a \in A$.

Schreier Internal Factorisation Systems

A subobject of morphisms $\alpha : A \to C_1$ of a Schreier internal category C in **Mon** is **Schreier** if for all $a \in A$, with $a = k_a + ed(a)$ for $k_a \in \mathrm{Ker}(d)$, we have that $k_a \in A$.

An internal factorisation system (ε, μ) on a Schreier internal category C in **Mon** is **Schreier** if ε and μ are Schreier.

Monoid Factorisation System

Let $\mathcal{X} = (X, +, 0)$ be a monoid, considered as a one object category. Let $(\mathcal{E}, \mathcal{M})$ be a (usual) factorisation system system on $\mathcal{X}$. Then:

1. $\mathcal{E}$ and $\mathcal{M}$ are submonoids of $\mathcal{X}$.
2. $\mathcal{E}$ and $\mathcal{M}$ contain all invertible elements of $\mathcal{X}$.
3. For all $x, y \in \mathcal{X}$, $e \in \mathcal{E}$, $m \in \mathcal{M}$ such that $y + e = m + x$, there exists a unique $z \in \mathcal{X}$ such that $z + e = x$ and $m + z = y$
4. For all $x \in \mathcal{X}$, there exists $e \in \mathcal{E}$ and $m \in \mathcal{M}$ such that $x = m + e$.

Schreier Internal Factorisation Systems

$\mathrm{SCat}(\textbf{Mon}) \quad \sim \quad \textbf{XSMod}$

$$\begin{tikzcd} C_0 \arrow[r, "e"] & C_1 \arrow[l, bend right, "d"'] \arrow[l, bend left, "c"] & C^{\leftarrow\leftarrow} \arrow[l, "m"'] \end{tikzcd} \quad \longleftrightarrow \quad (A, B, \alpha, f)$$

$$(\varepsilon, \mu) \quad \longleftrightarrow \quad (\mathcal{E}, \mathcal{M})$$

Schreier Internal Factorisation System

Monoid Factorisation System on A, with $\mathcal{E}$ and $\mathcal{M}$ closed under α

Future work

Does an internal factorisation system provide a reasonable definition for a factorisation system for double categories, viewed as objects of $\mathrm{Cat}(\mathbf{Cat})$?