Duality, unification, and admissibility in the positive fragment of Łukasiewicz logic

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- Infinite-valued Łukasiewicz logic introduced in 1930.
- It is a substructural logic: $\{\cdot, \to, \wedge, \vee, 0, 1\}$.
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- $[0,1]_{\mathsf{MV}}$ generates the variety of MV-algebras, the equivalent algebraic semantics of Łukasiewicz logic.
- Free MV-algebra $\mathcal{F}_{\mathsf{MV}}(n)$ = algebras of formulas over n-variables = algebras of McNaughton functions: $[0,1]^n \to [0,1]$.

The positive fragment of Łukasiewicz logic

- We consider the 0-free fragment of Łukasiewicz logic.
  Signature: $\{\cdot, \to, \wedge, \vee, 1\}$

- Since:
  $$x \wedge y = x \cdot (x \to y), \;\; x \vee y = (x \to y) \to y$$
  we consider the fragment in the language of hoops $\{\cdot, \to, 1\}$

- The equivalent algebraic semantics is the variety of Wajsberg hoops.

Hoops

Hoops introduced by Buchi and Owens, based on work by Bosbach.

A hoop is an algebra $\mathbf{A} = (A, \cdot, \to, 1)$ s.t. $(A, \cdot, 1)$ is a commutative monoid and:

(H1) $x \to x = 1$,

(H2) $x \cdot (x \to y) = y \cdot (y \to x)$,

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- $(\cdot, \to)$ form a *residuated pair* : $x \cdot y \leq z$ iff $y \leq x \to z$.
- Basic hoops (i.e., semilinear hoops) have a lattice order:

$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$$

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- Equivalent algebraic semantics of the positive fragment of Łukasiewicz logic Ł$^{+}$.
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- Fundamental to understand BL-algebras basic hoops [Aglianò, Montagna 2003].
- Known properties: deductive interpolation (via amalgamation, [Montagna 2006]), not structurally complete [Cintula Metcalfe 2009].
- The variety of Wajsberg hoops is generated by $[0,1]_{\mathsf{WH}} = ([0,1], \cdot_{Ł}, \to_{Ł}, 1)$

Free finitely-generated Wajsberg hoops can be characterized via McNaughton functions (Aglianò, Panti):

$$\mathcal{F}_{\mathsf{WH}}(n) = \{f \in \mathcal{F}_{\mathsf{MV}}(n) : f(\mathbf{1}) = 1\}$$

Alternative proof (sketch):

- $\mathcal{F}_{\mathsf{WH}}(n)$ is (isomorphic to) the 0-free subalgebra of positive terms in $\mathcal{F}_{\mathsf{MV}}(n)$.
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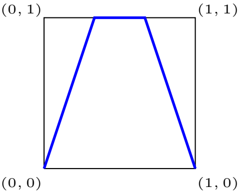
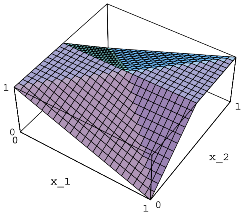
We use this to find a duality for finitely presented Wajsberg hoops.

Finitely presented in V: finitely generated quotient of a finitely generated free algebra.

(McNaughton's theorem): $\mathcal{F}_{\mathsf{MV}}(n)$ is the algebra of McNaughton's functions (piecewise linear with integer coefficients) from $[0,1]^n$ to $[0,1]$ with operations defined pointwise from $[0,1]_{\text{Ł}}$.





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$\phi : \neg(\neg x \oplus x^2)^3 \longmapsto f_\phi \in F_{MV}(1)$

$g \in F_{MV}(2)$

1-sets $\leftrightarrow$ rational polyhedra in $[0,1]^n$ (Mundici).

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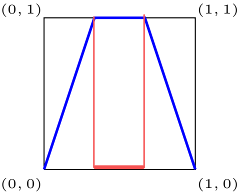

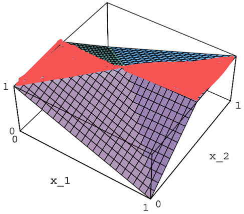

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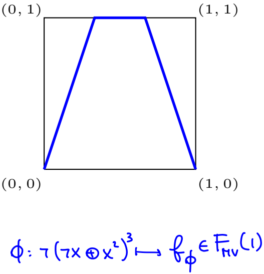

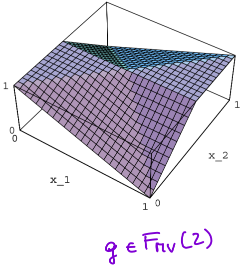

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Duality between finitely presented MV-algebras (with homomorphisms) and rational polyhedra (with $\mathbb{Z}$-maps: componentwise McNaughton's functions) (Marra, Spada).

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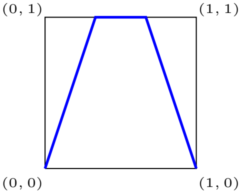

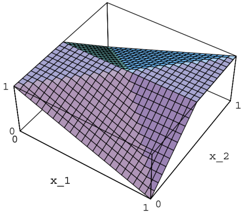

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Fin. gen quotient $\mathcal{F}_{\mathsf{MV}}(n)/\theta \leftrightarrow$ principal quotient $\mathcal{F}_{\mathsf{MV}}(n)/f \leftrightarrow$ 1-set of f

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- Given $f \in \mathcal{F}_{\mathsf{WH}}(n)$, the 1-set of f, O_f always contains $\mathbf{1}$.

- We show that finitely presented Wajsberg hoops are categorically equivalent to a (non-full) subcategory of finitely presented MV-algebras

- This category corresponds via Marra-Spada duality to pointed rational polyhedra with pointed $\mathbb{Z}$-maps:
  - objects: rational polyhedra P in $[0,1]^n$ such that $\mathbf{1} \in P$
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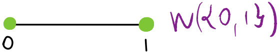
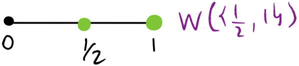
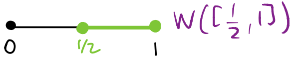
$$P \subseteq [0,1]^n \text{ pointed rational polyhedron} \longrightarrow \mathcal{W}(P) = \{f_{|P} : f \in \mathcal{F}_{\mathsf{WH}}(n)\}$$

Examples







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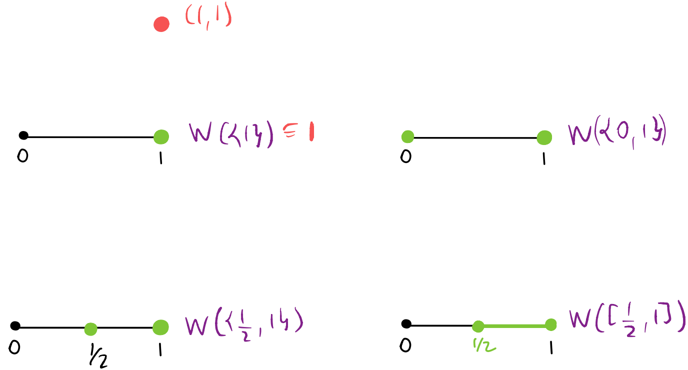

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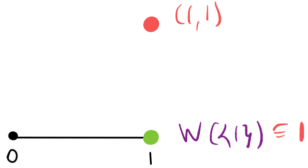

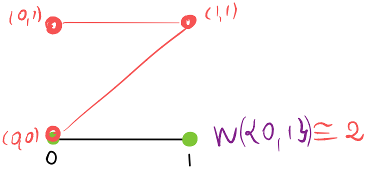

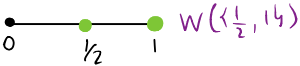

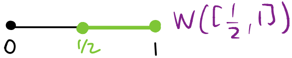

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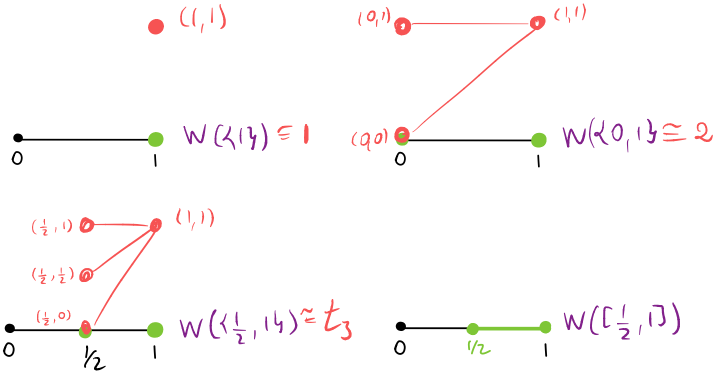

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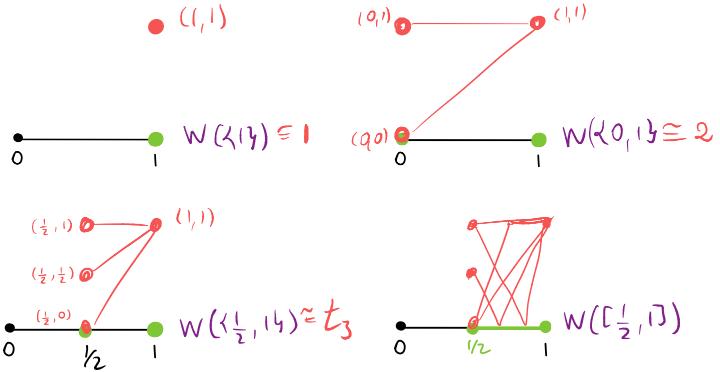

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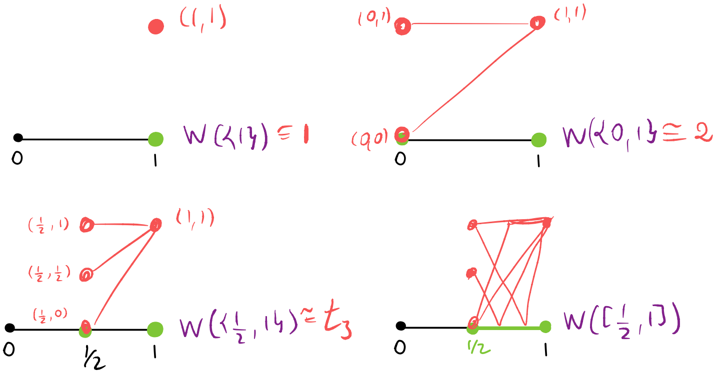

Proposition

Finitely presented subdirectly irreducible Wajsberg hoops are finitely generated subalgebras of [0, 1], and therefore bounded and simple.

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Let t, u be positive MV terms:

$$t \vdash_{\text{Ł}^+} u \quad \text{iff} \quad O_t \subseteq O_u \text{ iff } \quad t \vdash_{\text{Ł}} u$$

Unification via projectivity

Let $\mathcal{L}$ be an algebraizable logic, with equivalent algebraic semantics a variety V.

Unification problem: finite set of identities $\Sigma = \{s_i = t_i : i = 1 \ldots n\}$.

A **solution** or **unifier** is a substitution σ that makes the identities true in the variety: V $\models \sigma(s_i) = \sigma(t_i)$

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Projective algebras in a variety are retracts of free algebras:

$\mathbf{A}$ is a retract of $\mathbf{F}$ if $\mathbf{A} \underset{j}{\overset{i}{\rightleftarrows}} \mathbf{F}$, with $j \circ i = id$.

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Unification problem: finitely presented algebra **A** $\in$ V.
Solution or unifier: homomorphism u : **A** $\to$ **P**, **P** projective in V.

Unification via projectivity

Preorder on unifiers: $u_1 \leq u_2$ iff there exists $v : v \circ u_2 = u_1$.

The unification type (UT) of a unification problem can be: unitary, finitary, infinitary, or nullary, depending on the cardinality of maximal elements in the associated partial order (best solutions).

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Examples: Boolean algebras unitary [Balbes], Heyting algebras finitary [Ghilardi], Semigroups infinitary [Plotkin], MV-algebras nullary [Marra,Spada].

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Given $\mathbf{A}$ *Wajsberg hoop with* n *generators,* $\mathbf{A}$ *is projective iff* $\mathbf{A} \cong \mathcal{W}(P)$*, where* $P \subseteq [0,1]^n$ *is a retract of* $[0,1]^n$ *by a pointed* $\mathbb{Z}$*-map.*

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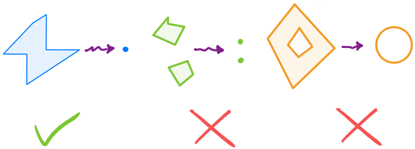

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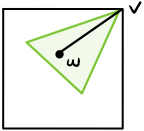
If P is a pointed $\mathbb{Z}$-retract of $[0,1]^n$ then

1. *P is contractible,*
2. *P is strongly regular.*



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We use the duality and works by [Cabrer, Cabrer Mundici] for the MV-algebraic framework.

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Given **A** *Wajsberg hoop with* n *generators,* **A** *is projective iff* $\mathbf{A} \cong \mathcal{W}(P)$, *where* $P \subseteq [0,1]^n$ *is a retract of* $[0,1]^n$ *by a pointed* $\mathbb{Z}$*-map.*

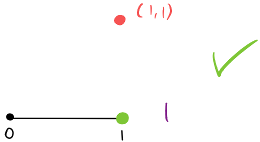
Theorem

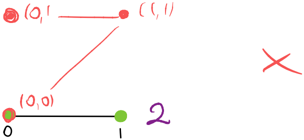
If P *is a pointed* $\mathbb{Z}$*-retract of* $[0,1]^n$ *then*

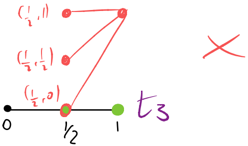
1. *P is contractible,*
2. *P is strongly regular.*

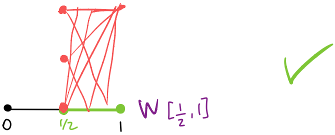
If P is one-dimensional, the converse also holds.

Examples

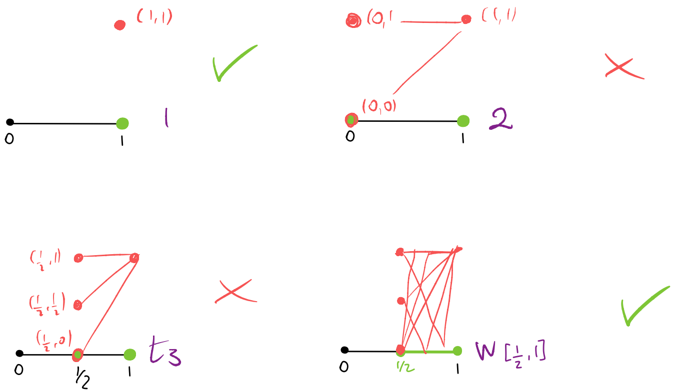








Examples

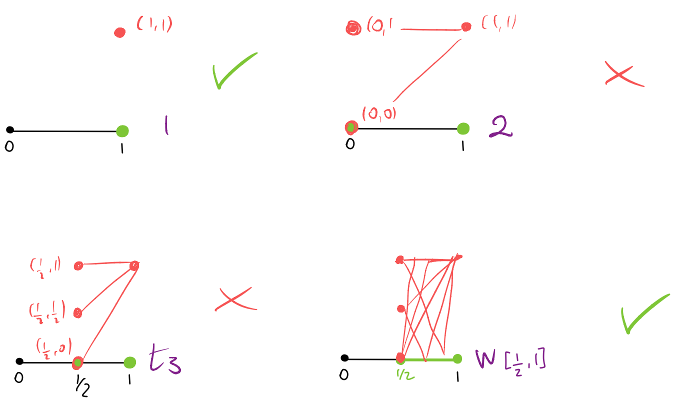


Proposition

No (nontrivial) bounded Wajsberg hoop is projective in a variety containing WH.

In particular: **2** is not projective in the variety of hoops or of residuated lattices.

Examples



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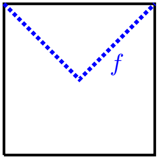
In particular: **2** is not projective in the variety of hoops or of residuated lattices.

Paolo Aglianò's talk: **2** is the only finite projective algebra in FL_{ew}.

Unification in WH

Unification problems in WH do not reduce to unification of 0-free terms in MV.



$f(x) = ((x \to x^2) \to x) \to x$

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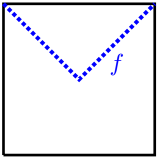

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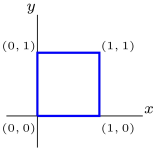
However: Marra and Spada pathological example to show that MV has nullary unification type can be adapted to Wajsberg hoops.

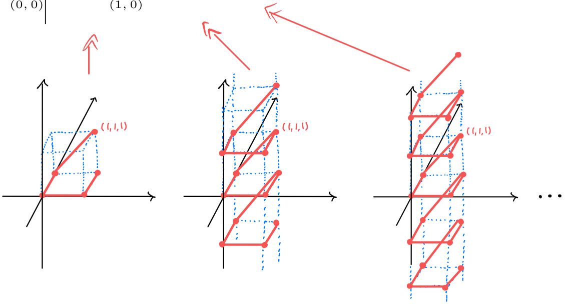
Unification in WH

Theorem

The unification type of Wajsberg hoops, and thus of the positive fragment of Łukasiewicz logic, is nullary.



$$[((x \to x^2) \to x) \to x] \vee [((y \to y^2) \to y) \to y] = 1$$



Admissibility

Let Σ, Δ be finite sets of identities in the language of a variety V.

A clause $\Sigma \Rightarrow \Delta$ is V-admissible if each unifier σ of Σ is also a unifier of some member of Δ.

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- the exact unification type of V is at most finitary
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Algebraically: generality order on coexact unifiers: onto homomorphisms from finitely presented algebras to exact algebras.

Exact algebra: finitely generated subalgebra of some finitely generated free algebra.

Ex: MV-algebras have nullary unification type but finitary exact type.

Exact Wajsberg hoops

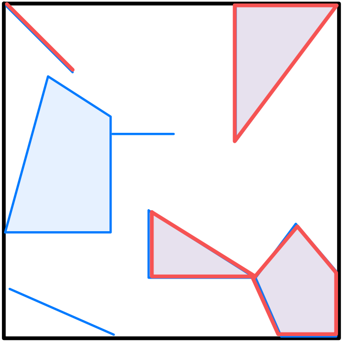
Via the duality and the work of Cabrer for the MV-algebraic framework:

Theorem

A Wajsberg hoop $\mathbf{A}$ *is exact iff* $\mathbf{A} \cong \mathcal{W}(P)$ *where* $P \subseteq [0,1]^n$ *is a pointed rational polyhedron that is connected and strongly regular.*

Admissibility in WH

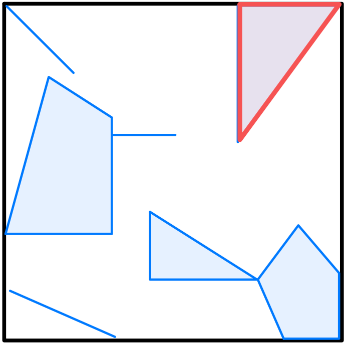
Given a finitely presented MV-algebra, one finds a **finite set** of maximal coexact unifiers ([Cabrer, Metcalfe],[Jeřábek]).



Admissibility in WH

Given a finitely presented MV-algebra, one finds a **finite set** of maximal coexact unifiers ([Cabrer, Metcalfe],[Jeřábek]).

Given a finitely presented Wajsberg hoop, one finds **one** maximal coexact unifier.



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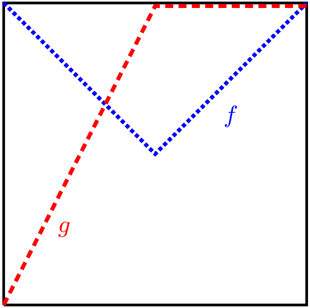
Admissibility in WH does not reduce to admissibility of 0-free MV-terms.



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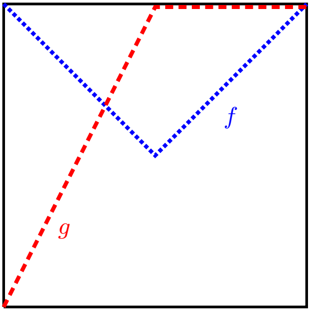

$f(x) = ((x \to x^2) \to x) \to x$

$g(x) = ((x \to x^2) \to x$

$(f = 1) \Rightarrow (g = 1)$ admissible in WH but not in MV.

Admissibility in WH

Wajsberg hoops have nullary unification type and unitary exact type.

Wajsberg hoops have the FEP (Blok Ferreirim), thus decidable equational theory.

Theorem

Admissibility of rules in Wajsberg hoops, and the positive fragment of Łukasiewicz logic, is decidable.

Conclusions

- Finitely presented Wajsberg hoops are dually equivalent to the category of pointed rational polyhedra
- Finitely generated projective and exact Wajsberg hoops can be characterized geometrically
- $Ł^+$ has nullary unification type, but unitary exact type, admissibility of rules is decidable.
- Future work: geometrical representation for finitely presented and projective BL-algebras and basic hoops?

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Thank you!

For more details and references: S. Ugolini, *The polyhedral geometry of Wajsberg hoops*, https://arxiv.org/abs/2201.07009.