On relative principal congruences in term quasivarieties

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Idea of this talk

Roughly speaking, in this talk we are going to study quasivarieties for which there exists a family of binary terms characterizing the relative principal congruences. This study is motivated by the fact that there are many quasivarieties of interest for algebraic logic for which the way to obtain a description of the relative principal congruences is exactly the same. As application we are going to mention some properties concerning relative compatible operations.

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Introduction (relative congruences)

Let A be an algebra.

- We write $\mathrm{Con}(A)$ by the poset of congruences of A.
- For $a, b \in A$, we write $\theta(a, b)$ by the principal congruence generated by the pair (a, b), i.e., the smallest congruence of A which contains the pair (a, b).

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- A congruence θ of A is said to be $\mathcal{K}$-congruence of A if $A/\theta \in \mathcal{K}$.
- We write $\mathrm{Con}_{\mathcal{K}}(A)$ for the poset of $\mathcal{K}$-congruences of A.
- We write $\theta_{\mathcal{K}}(a, b)$ by the $\mathcal{K}$-principal congruence generated by (a, b), i.e., the smallest $\mathcal{K}$-congruence of A which contains the pair (a, b) ($\mathrm{Con}_{\mathcal{K}}(A)$ is closed by arbitrary intersections).

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Remark

If $\mathcal{K}$ is a variety, $A \in \mathcal{K}$ and $a, b \in A$, then $\mathrm{Con}_{\mathcal{K}}(A) = \mathrm{Con}(A)$ and $\theta_{\mathcal{K}}(a, b) = \theta(a, b)$.

Introduction (relative compatible operations)

Let A be an algebra and $f : A^n \to A$ a function.

Definition

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- If $\mathcal{K}$ is a quasivariety and $A \in \mathcal{K}$, f is said to be $\mathcal{K}$-compatible if every $\mathcal{K}$-congruence of A is a $\mathcal{K}$-congruence of (A, f).

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In other words, f is $\mathcal{K}$-compatible if for every $a_1, b_1, \ldots, a_n, b_n \in A$ and $\theta \in \mathrm{Con}_{\mathcal{K}}(A)$, the following condition is satisfied:

If $(a_i, b_i) \in \theta$ for every $i = 1, \ldots\ n$ then $(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \theta$.

Introduction (relative compatible operations)

Remark

Let $\mathcal{K}$ be a quasivariety and $A \in \mathcal{K}$.

- Let $f : A^n \to A$ be a function and $\hat{a} = (a_1, \ldots, a_n) \in A^n$. For $i = 1, \ldots, n$ we define unary functions $f_i^{\hat{a}} : A \to A$ by

$$f_i^{\hat{a}}(b) := f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

  Then f is $\mathcal{K}$-compatible if and only if for every $\hat{a} \in A^n$ and $i = 1, \ldots, n$ it holds that $f_i^{\hat{a}}$ is $\mathcal{K}$-compatible.

- Let $f : A \to A$ be a function. Then f is $\mathcal{K}$-compatible if and only if for every $a, b \in A$ it holds that

$$(f(a), f(b)) \in \theta_{\mathcal{K}}(a, b).$$

Term quasivarieties (first the case of Heyting algebras)

Let $\mathcal{K}$ be the variety of Heyting algebras, $A \in \mathcal{K}$ and $a, b \in A$.

We define

$$a \leftrightarrow b := (a \to b) \wedge (b \to a).$$

It is known that

$$(x, y) \in \theta(a, b) \text{ if and only if } a \leftrightarrow b \leq x \leftrightarrow y.$$

We can may the following question:

How can we prove it?

Term quasivarities (first the case of Heyting algebras)

- Let $\theta \in \mathrm{Con}(A)$. Then

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- Let $\theta \in \mathrm{Con}(A)$. Then

$$(a, b) \in \theta \text{ if and only if } (a \leftrightarrow b, 1) \in \theta.$$

- There exists an order isomorphism between $\mathrm{Con}(A)$ and $\mathrm{Fil}(A)$ (the poset of filters of A), which is given by the assignments

$$\theta \mapsto 1/\theta,$$

$$F \mapsto \{(a, b) \in A \times A : a \leftrightarrow b \in F\}.$$

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Definition

For every $a \in A$ we define $[a) = \{b \in A : b \geq a\}$.

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Therefore

$$(x, y) \in \theta(a, b) \text{ iff } x \leftrightarrow y \in 1/\theta(a, b) \text{ iff } a \leftrightarrow b \leq x \leftrightarrow y.$$

Term quasivarieties

Let $\mathcal{K}$ be a quasivariety. Assume that the algebras of $\mathcal{K}$ have unless one operation of arity zero in the language. We choose the same constant for every member of $\mathcal{K}$, which will be denoted by e.
For every $A \in \mathcal{K}$ we define

$$\Sigma_{\mathbf{A}} = \{e/\theta : \theta \in \mathrm{Con}_{\mathcal{K}}(A)\}.$$

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Let $X \subseteq A$. We define

$$\langle X \rangle = \bigcap_{X \subseteq e/\theta} (e/\theta).$$

We have that $\langle X \rangle \in \Sigma_{\mathbf{A}}$ and this is the smallest element of $\Sigma_{\mathbf{A}}$ containing X. We say that $\langle X \rangle$ is the member of $\Sigma_{\mathbf{A}}$ generated by X.

Term quasivarieties

Definition

We say that $\mathcal{K}$ is a term quasivariety if there exist an operation of arity zero e and a family of binary terms $\{t_i\}_{i \in I}$ such that for every $A \in \mathcal{K}$, $\theta \in \mathrm{Con}_{\mathcal{K}}(A)$ and $a, b \in A$ the following property is satisfied:

$$(a, b) \in \theta \text{ if and only if } (t_i(a, b), e) \in \theta \text{ for every } i \in I.$$

In such case we say that $(e, \{t_i\}_{i \in I})$ is a pair associated to $\mathcal{K}$.

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If $\{t_i\}_{i\in I} = \{t\}$ we write (e, t) in place of $(e, \{t_i\}_{i\in I})$. If a term quasivariety $\mathcal{K}$ is a variety we also say that $\mathcal{K}$ is a term variety.

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Example

If $\mathcal{K}$ is the vatiety of Heyting algebras and $A \in \mathcal{K}$, for every θ congruence of A and $a, b \in A$ we have that $(a, b) \in \theta$ if and only if $(a \leftrightarrow b, 1) \in \theta$. Then $\mathcal{K}$ is a term variety where $(a \leftrightarrow b, 1)$ is a pair associated to $\mathcal{K}$.

Term quasivarieties: the main result

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Theorem

For every $A \in \mathcal{K}$ and $a, b, x, y \in A$,

$$(x, y) \in \theta_{\mathcal{K}}(a, b) \text{ if and only if } t_j(x, y) \in \langle \{t_i(a, b)\}_{i \in I} \rangle \text{ para cada } j \in I.$$

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Example

Consider $\mathcal{K}$ as the variety of Heyting algebras. Since we know that $\mathcal{K}$ is a term variety where $(a \leftrightarrow b, 1)$ is a pair associated to $\mathcal{K}$ we get

$$(x, y) \in \theta(a, b) \text{ iff } x \leftrightarrow y \in \langle a \leftrightarrow b \rangle \text{ iff } a \leftrightarrow b \leq x \leftrightarrow y.$$

Term quasivarieties: some questions

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  The answer is yes. It can be showed that the variety of distributive lattices with top is not a term variety.

- Consider the variety of distributive lattices with top. Does this variety satisfy the conclusion of the theorem?

  The answer is negative.

Some examples of term quasivarieties (varieties)

Quasivariety	$\{t_i(a,b)\}_{i\in I}$	$\Sigma_{\mathbf{A}}$	$\langle\{t_i(a,b)\}_{i\in I}\rangle$
1) Conm. res. lattices	$\{s(a,b)\}$	Convex sub.	It is known
2) Hilbert algebras	$\{a\to b, b\to a\}$	Imp. filters	It is known
3) BCK-algebras	$\{a\to b, b\to a\}$	Imp. filters	It is known

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$$(x,y)\in\theta_{\mathcal{K}}(a,b) \text{ iff } \ldots$$

1) $s(x,y)\in\langle\{s(a,b)\rangle$ iff $\exists n$ tq $s(a,b)^n\leq s(x,y)$, where

$$s(a,b) := ((a\to b)\wedge e)\cdot((b\to a)\wedge e).$$

2) $x\to y, y\to x\in\langle\{a\to b, b\to a\}\rangle$ iff $a\to b\leq(b\to a)\to(x\to y)$ and $a\to b\leq(b\to a)\to(y\to x)$.

3) $x\to y, y\to x\in\langle\{a\to b, b\to a\}\rangle$ iff $\exists n$ st. $\ldots$

Other examples of term quasivarieties

- Residuated lattices
- Pseudo BCK-algebras
- Semi-Heyting algebras
- Implicative semilattices
- ...

$\mathcal{K}$-compatible operations

In what follows we write $\mathcal{K}$ for a term quasivariety where $(e, \{t_i\}_{i \in I})$ is a pair associated to $\mathcal{K}$.

Proposition

Let $f : A^k \to A$ be a function. The following conditions are equivalent:

- f is $\mathcal{K}$-compatible.
- For every $\hat{a} \in A^k$, $x, y \in A$ and $l = 1, \ldots, k$,

$$(f_l^{\hat{a}}(x), f_l^{\hat{a}}(y)) \in \theta_{\mathcal{K}}(x, y).$$

- For every $\hat{a} \in A^k$, $x, y \in A$, $j \in I$ and $l = 1, \ldots, k$,

$$t_j(f_l^{\hat{a}}(x), f_l^{\hat{a}}(y)) \in \langle \{t_i(x, y)\}_{i \in I} \rangle.$$

Example: the variety of Heyting algebras

Let $f : A^2 \to A$ be a function. The following conditions are equivalent:

- f is compatible.
- For every $x_1, x_2, y_1, y_2 \in A$,

$$x_1 \leftrightarrow y_1 \leq f(x_1, x_2) \leftrightarrow f(y_1, x_2),$$

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The second cond. and the inequality $(a \to b) \wedge (b \to c) \leq a \to c$ implies

$$(x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \leq f(x_1, x_2) \leftrightarrow f(y_1, y_2). \tag{1}$$

The inequality (1) for the cases i) $y_2 = x_2$, ii) $x_1 = y_1$ give us the second condition.

$\mathcal{K}$-compatible operations

Possible applications of the description of $\mathcal{K}$-compatible functions:

- Suppose that in the algebras of $\mathcal{K}$ it can be defined an order. We can give necessary conditions on $\mathcal{K}$ for which for every $A \in \mathcal{K}$ the $\mathcal{K}$-compatible operations on A are equal to a supremum of polynomials in each finite subset of A. Under these conditions, if $\mathcal{K}$ is a variety whose algebras have supremum in the language (associated to the order of the algebras), then the variety $\mathcal{K}$ is locally affine complete (i.e., every compatible operation is equal to a polynomial in each finite subset).
- We can give methods in order to build up $\mathcal{K}$-compatible operations.

Some references

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