Regular algebras over semimonads

Ülo Reimaa

University of Tartu

Université catholique de Louvain

24.06.2022

(Semi)monads

An endofunctor $T : \mathcal{C} \to \mathcal{C}$ of a category is called a **monad**, if it is accompanied by natural transformations

$$\mu \colon TT \to T \quad \text{and} \quad \eta \colon 1_{\mathcal{C}} \to T$$

which make the diagrams

$$\begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TT \arrow[d, "T"] \\ TT \arrow[r, "T"'] & T \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} T \arrow[r, "\eta 1"] \arrow[d, "1"'] & TT \arrow[dl, "\mu"'] \\ T \arrow[r, "1"'] & T \arrow[u, "1\eta"'] \end{tikzcd}$$

commute.

If the unit $\eta \colon 1 \to T$ is not included, we get a **semimonad**. (Could also be called a **semigroupad**.)

Algebras over a (semi)monad

An **algebra** over a semimonad $T : \mathcal{C} \to \mathcal{C}$ is an object A of $\mathcal{C}$ along with a morphism $TA \to A$ such that the diagram

$$\begin{tikzcd} TTA \arrow[r, "T\xi_A"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\xi_A"] \\ TA \arrow[r, "\xi_A"'] & A \end{tikzcd}$$

commutes. If T is a monad, we would also ask for the diagram

$$\begin{tikzcd} A \arrow[d, "\eta_A"'] \arrow[dr, "1"] & \\ TA \arrow[r, "\xi_A"'] & A \end{tikzcd}$$

to commute. In the latter case we have an inclusion

$$\mathsf{Alg}(T, \mu, \eta) \to \mathsf{Alg}(T, \mu)\,.$$

Examples of algebras over a monad

Example

The functor $T\colon \mathrm{Set} \to \mathrm{Set}$ mapping a set T to the free group FX (viewed as a set), is a monad.

A T-algebra is a group G, along with the map $\xi_G\colon TG \to G$, which maps a group word, such as aba^{-1}, to its evaluation in G.

Example

If M is a monoid, then we have a monad

$$- \times M\colon \mathrm{Set} \to \mathrm{Set}\,.$$

Its algebras are M-sets, meaning sets X along with an action

$$\xi_M\colon X \times M \to X$$

of M on X, satisfying $(xm)m' = x(mm')$ and $x1 = x$.

Examples of algebras over a semimonad

Example

If S is a semigroup, then we have a semimonad

$$- \times S \colon \mathsf{Set} \to \mathsf{Set}$$

whose algebras are S-sets, meaning sets X along with an action

$$\xi_M \colon X \times S \to X$$

satisfying $(xs)s' = x(ss')$.

Example

The above also works for any semigroup object S in a monoidal category $\mathcal{V}$, giving the semimonad $- \otimes S$ on $\mathcal{V}$.

Taking $\mathcal{V}$ to be abelian groups (or sup-lattices), the algebras will be modules over a non-unital ring (or a non-unital quantale).

$\mathrm{Alg}(T, \mu)$ vs $\mathrm{Alg}(T, \mu, \eta)$

If T is a monad, there is an easy way of identifying which objects of $\mathrm{Alg}(T, \mu)$ belong to $\mathrm{Alg}(T, \mu, \eta)$.

Proposition

Let (T, μ, η) be a monad and suppose that $\xi_A \colon TA \to A$ is an algebra over the semimonad (T, μ). Then the following statements are equivalent:

1. *$\xi_A \colon TA \to A$ is an epimorphism,*
2. *$\xi_A \colon TA \to A$ is a split epimorphism,*
3. *the following is a coequalizer diagram*

$$TTA \underset{T\xi_A}{\overset{\mu_A}{\rightrightarrows}} TA \xrightarrow{\xi_A} A \ ,$$

4. *$\xi_A \colon TA \to A$ is an algebra over the monad (T, μ, η).*

$\mathrm{Alg}(T, \mu)$ vs $\mathrm{Alg}(T, \mu, \eta)$

1. $\xi_A \colon TA \to A$ is an epimorphism and a (T, μ)-algebra,
4. $\xi_A \colon TA \to A$ is an algebra over the monad (T, μ, η).

(1) $\implies$ (4).

$$
\begin{tikzcd}
TA \arrow[rrr, "\xi_A"] \arrow[dr, "\eta_{TA}"] \arrow[dd, "1"'] & & & A \arrow[dl, "\eta_A"'] \arrow[dd, "1"] \\
& TTA \arrow[r, "T\xi_A"] \arrow[dl, "\mu_A"'] & TA \arrow[dr, "\xi_A"] & \\
TA \arrow[rrr, "\xi_A"] & & & A
\end{tikzcd}
$$

$\square$

Regular algebras over a semimonad

Suppose now that T is an arbitrary semimonad for which there doesn't necessarily exist an $\eta\colon 1 \to T$ making T into a monad.

We can view T as an endofunctor of $\mathsf{Alg}(T)$, allowing us to view

$$TTA \underset{T\xi_A}{\overset{\mu_A}{\rightrightarrows}} TA \xrightarrow{\xi_A} A$$

as a diagram of T-algebras.

Definition

Let us say that an algebra $\xi_A\colon TA \to A$ over a semimonad T is **regular** if

$$TTA \underset{T\xi_A}{\overset{\mu_A}{\rightrightarrows}} TA \xrightarrow{\xi_A} A$$

is a coequalizer diagram in $\mathsf{Alg}(T)$.

(For simplicity, we can assume that T preserves coequalizers.)

If T does not preserve coequalizers

Disclaimer

If T does not preserve coequalizers, you can take the precise definitions in this presentation with a grain of salt.

Some of the results need certain coequalizers involving the maps μ_A and $T\xi_A$ to be preserved or reflected by the functors between $\mathcal{C}$ and $\mathrm{Alg}(T)$.

If T preserves coequalizers, then everything works, but if T doesn't preserve all coequalizers, it is a work in progress to determine what the correct baseline assumptions should be.

Representable algebras

Given a semimonad T on $\mathcal{C}$, we have the functor

$$F^T : \mathcal{C} \to \mathsf{Alg}(T),$$

which maps an object A of $\mathcal{C}$ to the algebra

$$\mu_A \colon TTA \to TA$$

on the object TA.

Note that this TA is generally not the free algebra on A, so let us instead call the algebras TA the **representable algebras**.

Asking an algebra A to be regular amounts to asking for the algebra to be expressible as a canonical coequalizer

$$TTA \underset{T\xi_A}{\overset{\mu_A}{\rightrightarrows}} TA \xrightarrow{\xi_A} A$$

of representable algebras.

Regular semimonads

In a lot of what follows, we want the representable algebras to behave well. At the very least, we would like for them to be regular, allowing us to consider the representable algebra functor as a functor

$$F^T \colon \mathcal{C} \to \mathrm{RegAlg}(T)\,.$$

Definition

Let us say that a semimonad T is **regular**, if all the diagrams

$$TTTA \overset{\mu_{TA}}{\underset{T\mu_A}{\rightrightarrows}} TTA \xrightarrow{\mu_A} TA$$

are coequalizers in $\mathcal{C}$ and $\mathrm{Alg}(T)$.

Adjunctions and monads

For each adjoint pair of functors $F \dashv G$ we have the diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[rr] \arrow[dr, bend left, "G"] & & \mathrm{Alg}(T)\,, \arrow[dl, bend left, "U"] \\
& \mathcal{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] &
\end{tikzcd}$$

where the endofunctor $T = GF$ of $\mathcal{C}$ carries a monad structure.

For any monad T, the free algebra functor F^T is left adjoint to the forgetful functor U.

Copointed endofunctors and semimonads

An endofunctor $H\colon \mathcal{C} \to \mathcal{C}$ is said to be **copointed** if it is accompanied by a natural transformation

$$\chi\colon H \to 1_{\mathcal{C}}\,.$$

For pair of functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ along with a copointed structure $FG \to 1$ on the composite FG, the functor $T = GF$ is a semimonad and we have the diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "K"] \arrow[dr, bend left, "G"] & & \mathrm{Alg}(T)\,. \arrow[dl, bend left, "U"] \\
& \mathcal{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] &
\end{tikzcd}$$

For any semimonad T, we have a pointed structure on $F^T U$, given by the algebra structure maps $F^T UA = TA \xrightarrow{\xi_A} A$.

Left adjoint to the comparison functor (monad case)

The left adjoint $Q\colon \mathrm{Alg}(T) \to \mathcal{D}$ maps an algebra $\xi_A\colon TA \to A$ to the coequalizer

$$\begin{tikzcd} FGFA \arrow[r, shift left, "F\xi_A"] \arrow[r, shift right, "\varepsilon_{FA}"'] & FA \arrow[r] & Q(A,\xi) \end{tikzcd} .$$

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "K"'] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[dr, bend left, "G"] & & \mathrm{Alg}(T) \arrow[ll, bend right, "Q"'] \arrow[dl, bend left, "U"] \\ & \mathcal{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] & \end{tikzcd}$$

Here KA is the T-algebra

$$G\varepsilon_A\colon GFGA \to GA\,.$$

$$\begin{tikzcd} GFGFA \arrow[r, shift left, "GF\xi_A"] \arrow[r, shift right, "G\varepsilon_{FA}"'] & GFA \arrow[r] \arrow[dr, "\xi_A"'] & GQ(A,\xi) \arrow[d, dashed] \\ & & A \end{tikzcd}$$

The counit of the adjunction $Q \dashv K$ exists because ξ_A is a coequalizer of the pair of arrows on the left.

Left adjoint to the comparison functor (semimonad case)

To do the same in the case of semimonads, we need to replace $\mathrm{Alg}(T)$ with $\mathrm{RegAlg}(T)$:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "K"'] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[dr, bend left, "G"] & & \mathrm{RegAlg}(T) \arrow[ll, bend right, "Q"'] \arrow[dl, bend left, "U"] \\
& \mathcal{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] &
\end{tikzcd}$$

Additionally we have to assume that:

1. The functor F preserves coequalizers and that
2. for each object A of $\mathcal{D}$ the diagram

$$GFGFGA \underset{G\varepsilon_{FG_A}}{\overset{GFG\varepsilon_A}{\rightrightarrows}} GFGFA \xrightarrow{R\varepsilon_A} GA$$

is a coequalizer.

Reflection of algebras into regular algebras

$$TTA \overset{\mu_A}{\underset{T\xi_A}{\rightrightarrows}} TA \xrightarrow{\xi_A} A$$

The above need not always be a coequalizer diagram in $\mathrm{Alg}(T)$, but if it isn't, it makes sense to consider the actual coequalizer.

Define $\mathcal{T}A$ as the coequalizer

$$TTA \overset{\mu_A}{\underset{T\xi_A}{\rightrightarrows}} TA \xrightarrow{\omega_A} \mathcal{T}A \ .$$

This is a coequalizer in the category of endofunctors of $\mathrm{Alg}(T)$, so $\mathcal{T}$ will be an endofunctor of $\mathrm{Alg}(T)$.

Reflection of algebras into regular algebras

The universal property of the coequalizer gives us a comparison morphism

$$\begin{tikzcd} TTA \arrow[r, shift left, "\mu_A"] \arrow[r, shift right, "T\xi_A"'] & TA \arrow[r, "\omega_A"] \arrow[dr, "\xi_A"'] & \mathcal{T}A \arrow[d, dashed, "\tau_A"] \\ & & A\,. \end{tikzcd}$$

The morphism $\tau_A \colon \mathcal{T}A \to A$ is an isomorphism precisely when A is a regular algebra.

Furthermore, if T is a regular semimonad, the algebras $\mathcal{T}A$ are regular and

$$\mathcal{T} \colon \mathsf{Alg}(T) \to \mathsf{RegAlg}(T)$$

is the coreflection of $\mathsf{Alg}(T)$ into $\mathsf{RegAlg}(T)$.

Adjoint semimonads

Let (L, μ) be a semimonad such that the functor $L\colon \mathcal{C} \to \mathcal{C}$ has a right adjoint R.

Then $\mu\colon LL \to L$ has a mate $\nu\colon R \to RR$ which makes (R, ν) into a cosemimonad.

Furthermore, the correspondence

$$LA \to A \quad \Leftrightarrow \quad A \to RA$$

is an isomorphism of categories between $\mathrm{Alg}(L)$ and $\mathrm{CoAlg}(R)$.

If L is a regular semimonad, then R is a coregular cosemimonad, and the category of regular L-algebras is equivalent to the category of coregular R-coalgebras.