From $\{0,1\}$ to $[0,1]$: A survey of duality theorems

Vincenzo Marra — Dipartimento di Matematica “Federigo Enriques”, Milan, Italy

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Lawvere theories on the finite powers of the set $\{0,1\}$

$$T \;:\; \text{objects } \{0,1\}^n \;,\; n \in \mathbb{N}, \text{ and morphisms}$$

$$\mathrm{Alg}\, T \;:\; \text{Functors } T \to \mathsf{Set} \text{ preserving finite products}$$

$$T \;\cong\; (\text{Free finitely generated } \mathrm{Alg}\, T)^{\mathrm{op}}$$

Choice of morphisms determines structure on (powers of) $\{0,1\}$

1. All functions (sets). Theory of Boolean algebras.
2. All monotone functions (posets). Theory of distributive lattices.

Stone Duality, I

Theory: All functions. Algebras: Boolean algebras.

$$\text{Finite sets } 2^n \;\cong\; (\text{Free finitely generated B.A.})^{\mathrm{op}}$$

$$\Big\downarrow$$

$$\text{Stone Spaces} \cong (\text{Boolean Algebras})^{\mathrm{op}}$$

The lifting $\downarrow$ is a form of completion, and can be explained by general theory in which Ind/Pro-completions play a key rôle. But obtaining a satisfactory description of the resulting category on either side may be highly challenging.

Stone Duality, II

Theory: Monotone functions. Algebras: Distributive lattices.

$$\text{Finite posets } 2^n \;\cong\; (\text{Free finitely generated D.L.})^{\text{op}}$$

$$\downarrow$$

$$\text{Priestley Spaces} \cong (\text{Distributive Lattices})^{\text{op}}$$

Alternatively, replace:

- Finite posets 2^n with Finite Alexandrov spaces 2^n
- Monotone maps with Continuous (spectral) maps
- Priestley Spaces with Spectral Spaces

Lawvere-Linton theories on powers of the set $[0,1]$

$$T \ : \ \text{objects } [0,1]^{\kappa} \ , \ \kappa < \lambda, \ \text{and morphisms}$$

$$\mathrm{Alg}\, T \ : \ \text{Functors } T \to \mathsf{Set} \text{ preserving all } \kappa\text{-ary products}$$

$$T \cong (\text{Free } \kappa\text{-generated } \mathrm{Alg}\, T)^{\mathrm{op}}$$

Choice of morphisms determines structure on (powers of) $[0,1]$

Lifting to all algebras is again by (adapted) general theory

Unmotivated choice of morphisms is likely uninteresting

Above continuity arity is countable

$$\begin{tikzcd} {[0,1]^I} \arrow[r, "f"] \arrow[d, dashed, "\pi"'] & {[0,1]} \\ {[0,1]^{\mathbb{N}}} \arrow[ur, dashed] & \end{tikzcd}$$

By the Stone-Weierstrass Theorem, the continuous function f depends on at most a countable infinity of coordinates, because it can be written as the uniform limit of a sequence of functions each depending on finitely many coordinates (e.g., polynomials).

Continuous functions: Stone-Gelfand Duality

- T : Tychonoff cubes $[0,1]^n$, $[0,1]^{\mathbb{N}}$, and all continuous maps.
- Free algebras on I : $\mathrm{C}\left([0,1]^I\right)$, continuous functions to $[0,1]$.
- Simple algebras: $[0,1]$ (initial).
- $\operatorname{Alg} T = \mathbb{SP}\,[0,1]$ (in particular, semisimplicity).
- E.g., $\operatorname{Alg} T$ has no proper non-trivial subvariety.
- $\operatorname{Alg} T$: $\mathrm{C}\,(X)$, X compact Hausdorff.
- Duality : $(\operatorname{Alg} T)^{\mathrm{op}} \cong \mathsf{KH}$, compact Hausdorff spaces.

VM and L. Reggio, *Stone duality above dimension zero: Axiomatising the algebraic theory of C(X)*, Adv. Math., 2017

Primitive operations, and Axiomatisation

- $\neg x := 1 - x$; and $x \oplus y := \min\{x + y, 1\}$, 0 — MV-algebras
- Isbell's operation

$$\delta(x_1, x_2, \ldots) := \sum_{i \geqslant 1} \frac{x_i}{2^i} : [0,1]^{\mathbb{N}} \longrightarrow [0,1]$$

- A dozen equations in the signature $\neg, \oplus, 0, \delta$ axiomatise $\mathrm{Alg}\, T$.
- E.g. writing $\frac{1}{2}(x)$ for $\delta(x, 0, 0, \ldots)$:

$$\delta(x, x, \ldots) = x$$

$$\frac{1}{2}(\delta(x_1, x_2, \ldots) = \delta(\frac{1}{2}(x_1), \frac{1}{2}(x_2), \ldots)$$

So $\mathrm{Alg}\, T$ is the finitely-axiomatised variety of these "δ-algebras"

Logic

- Birkhoff used infinitary operations through the mid-Forties.
- 1944: Subdirect Representation requires finitarity ($\mathbb{HSP}$ Theorem does not).
- There seems to be relatively little work on "infinitary algebraic logic".
- Model-theoretic logics (Mostowski, Tarski, Lindström) use different, non-algebraic sort of infinitarity—and:
- $\mathsf{KH}^{\mathrm{op}}$ is not the category of models of a first-order theory (cf. also S. Henry, JSL, 2019):

  M. Lieberman, J. Rosicky, S. Vasey, *Hilbert spaces and C*-algebras are not finitely concrete*, arXiv.1908.10200.v5

- This, I think, lends further support to developing "infinitary algebraic logic", as in:

Logic

- Reggio studied the equational consequence relation associated with the variety of δ-algebras.
- Obtained a Hilbert-style correspondent.
- Proved a (strong) Completeness Theorem.
- Proved a Beth Definability Theorem and showed it is equivalent to the Stone-Weierstrass Theorem.

L. Reggio, *Beth definability and the Stone-Weierstrass Theorem*, Ann. Pure Appl. Logic, 2021

Monotone functions: Nachbin Duality

Next regard [0,1] as ordered.

A *compact ordered space* is a compact (Hausdorff) space X with a partial order that is closed in the product $X \times X$. Morphisms of $\mathsf{KH}_{\leqslant}$: continuous monotone maps.

D. Hofmann, R. Neves, and P. Nora, *Generating the algebraic theory of C(X): the case of partially ordered compact spaces*, Theory App. Categ. 33, 2018

M. Abbadini, *The dual of compact ordered spaces is a variety*, Theory App. Categ. 34, 2019

M. Abbadini, L. Reggio, *On the axiomatisability of the dual of compact ordered spaces*, Appl. Categ. Structures 28, 2020

Monotone functions: Nachbin Duality

- T : Nachbin cubes $[0,1]^n$, $[0,1]^{\mathbb{N}}$, and all monotone continuous maps.
- Free algebras on I : $\mathrm{C}_{\leqslant}([0,1]^I)$, monotone continuous functions to $[0,1]$.
- $\mathrm{Alg}\, T = \mathbb{SP}\,[0,1]$.
- $\mathrm{Alg}\, T$: $\mathrm{C}_{\leqslant}(X)$, X a compact ordered space.
- Duality : $(\mathrm{Alg}\, T)^{\mathrm{op}} \cong \mathsf{KH}_{\leqslant}$, compact ordered spaces.

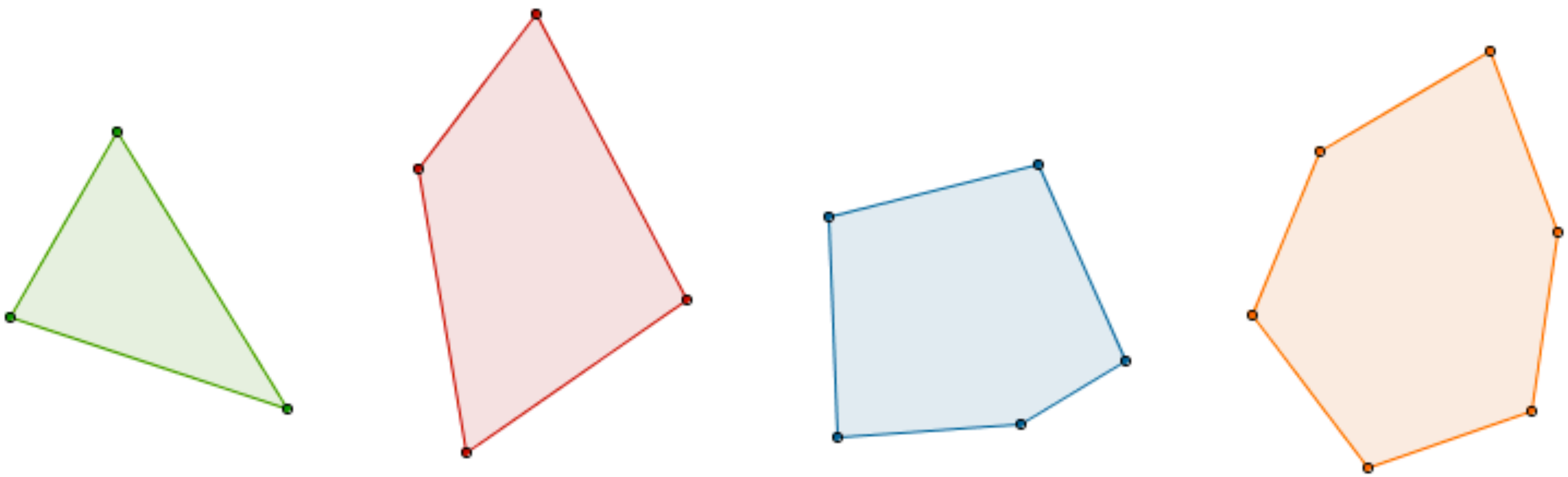
Primitive operations, and Axiomatisation

- Abbadini obtains finite equational axiomatisation of $\mathrm{Alg}\,T$.
- Builds on dual adjunction between a finitary variety **MC** of "monotone (negation-free) MV-algebras", and $\mathsf{KH}_{\leqslant}$.
- Uses ordered versions of Urysohn's Lemma and of the Stone-Weierstrass Theorem to characterise induced dual equivalence.
- Axiomatises the finitary algebras fixed by the dual adjunction using an infinitary operation—in analogy to Isbell's—that maps *sufficiently many* Cauchy sequences to their limits.

So $\mathrm{Alg}\,T$ is the finitely-axiomatised variety of these "MC_{∞}-algebras"

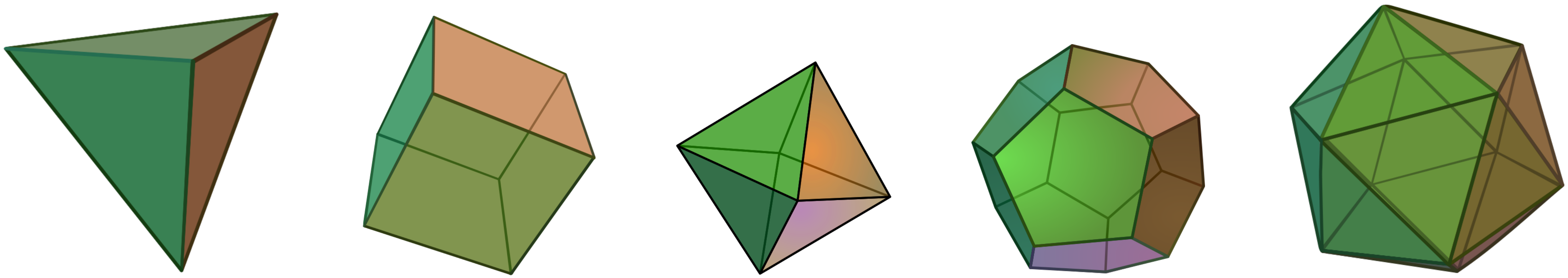
Piecewise-affine functions: Baker-Beynon Duality

Polytopes are convex hulls of finite sets of points.



Some polygons in the plane

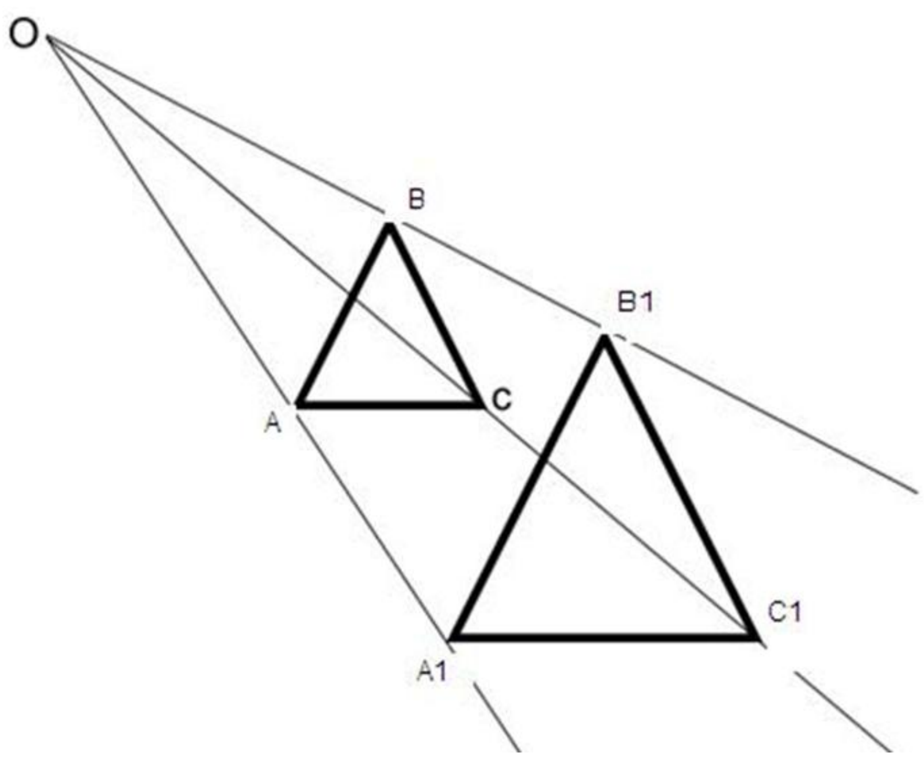
Cubes $[0,1]^n$ are polytopes

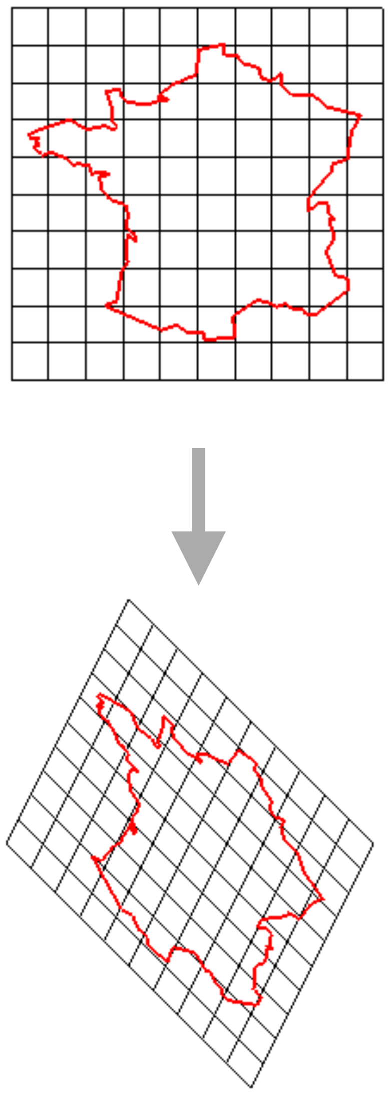


The Platonic solids

(After Plato's "Timaeus", ca. 350 BC.)

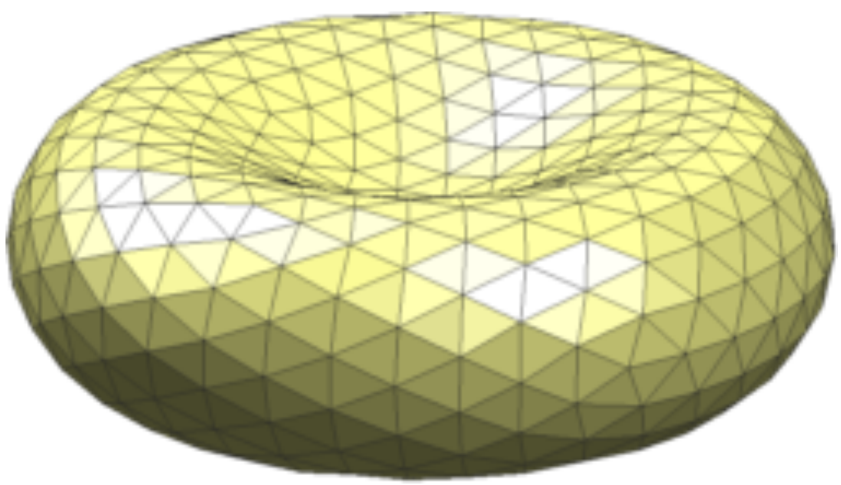
Morphisms between polytopes are affine maps

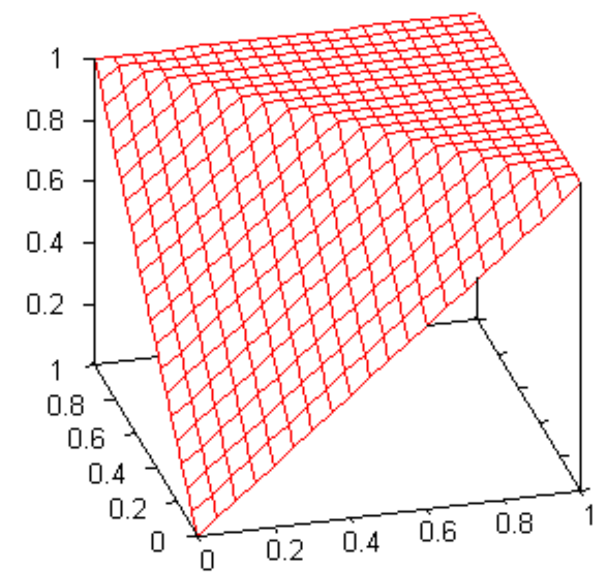




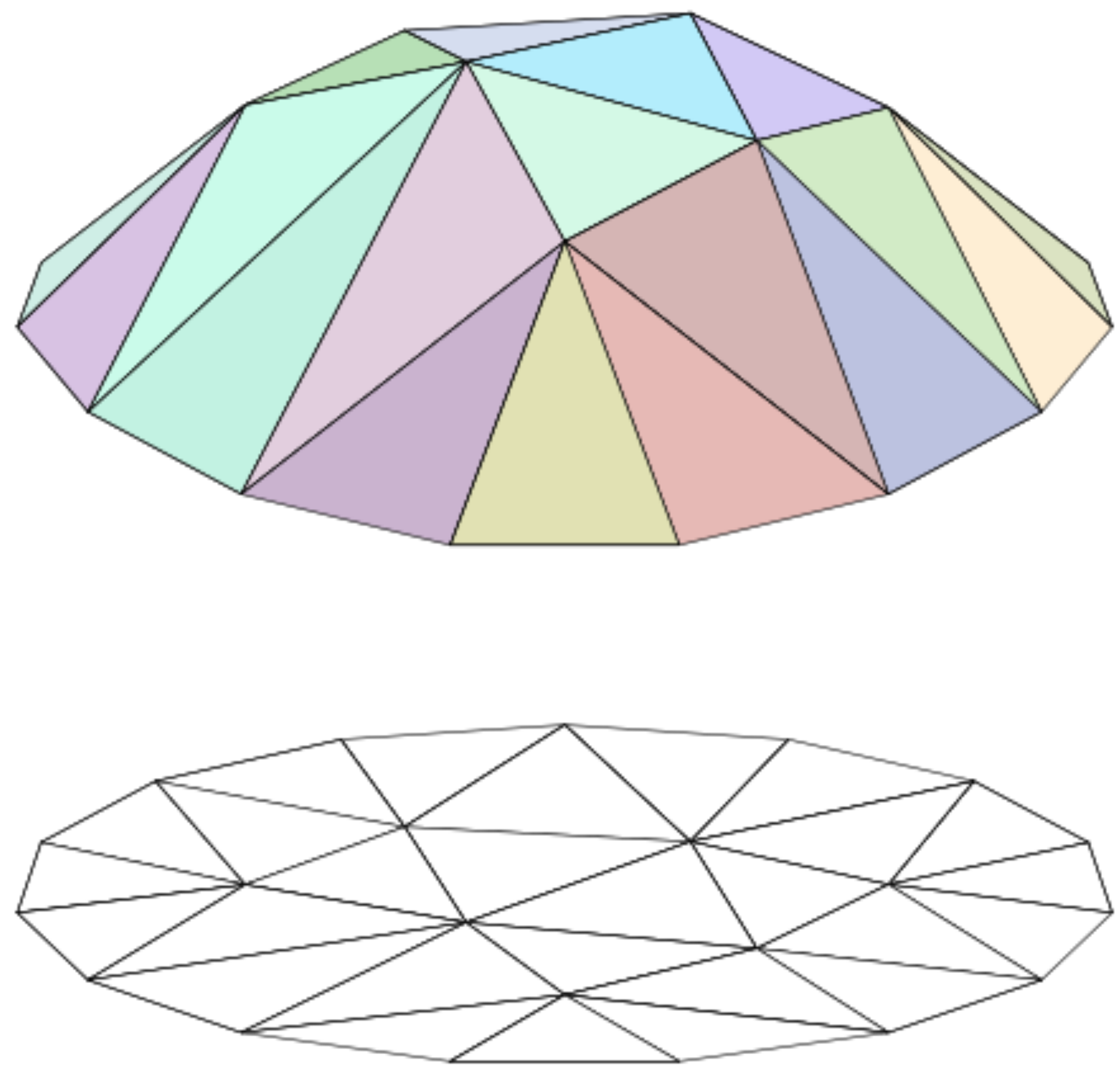
Affine maps are linear transformations with respect to an arbitrarily chosen origin. They don't preserve distance, angles or orientation, but they do preserve *parallelism* and *convexity*.

Polyhedra are finite unions of polytopes





Morphisms between polyhedra are piecewise-affine (PL) maps



Graph of a PL function is a polyhedron. PL maps are obtained by gluing together finitely many affine maps, just like polyhedra are obtained gluing together finitely many polytopes. So PL maps are "locally affine maps" (with finitely many affine pieces).

PL functions: Baker-Beynon Duality

- T : Cubes $[0,1]^n$, and all PL maps.
- Free algebras on I : $\nabla([0,1]^I)$, PL functions to $[0,1]$.
- Simple algebras: $[0,1]$ (initial).
- $\mathrm{Alg}\, T = \mathbb{HSP}\,[0,1]$ – $\mathbb{H}$ yields non-Archimedean objects.
- $(\mathrm{Alg}\, T)_{\mathrm{fp}}$: $\nabla(P)$, P a compact polyhedron.
- Duality : $(\mathrm{Alg}\, T)^{\mathrm{op}}_{\mathrm{fp}} \cong \mathsf{P}$, category of compact polyhedra.

It is possible to give an explicit (uncountable!) equational axiomatisation of $\mathrm{Alg}\, T$ — let's take a non-equational detour instead.

PL functions: Baker-Beynon Duality

A *unital vector lattice* is a lattice-ordered real linear space with a distinguished element (the unit) whose multiples eventually dominate any given element.

Yosida's 1941 version of Stone-Gelfand Duality represents $\mathsf{KH}^{\mathrm{op}}$ as the category of such unital vector lattices with the property that the pseudo-metric induced by the unit is a metric, and the structure is Cauchy complete in this metric. Morphisms are unit-preserving linear lattice homomorphisms.

In its obvious finitary algebraic signature (lattice, real linear space, unit), Yosida's category is not elementary—for general reasons, as we've seen.

PL functions: Baker-Beynon Duality

Restrict to the full subcategory $\mathbf{UVL}_{\mathrm{fp}}$ of (Gabriel-Ulmer) finitely presentable objects.

Beynon (1977), building on Baker (1968), proved:

$\mathbf{UVL}_{\mathrm{fp}}^{\mathrm{op}} \cong \mathbf{P}$, the category of compact polyhedra.

Returning to the equational setting, we thus have:

$$(\mathrm{Alg}\, T)_{\mathrm{fp}} \cong \mathrm{UVL}_{\mathrm{fp}}.$$

This equivalence may be explicitly described using the Chang-Mundici theory of unit intervals in lattice-groups.

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- Duality : $(\mathrm{Alg}\, T)_{\mathrm{fp}}^{\mathrm{op}} \cong \mathsf{P}$, category of compact polyhedra.

The end result is an explicit (but uncountable!) equational axiomatisation of $\mathrm{Alg}\, T$.

PL functions: Baker-Beynon Duality

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- Duality : $(\mathrm{Alg}\,T)^{\mathrm{op}}_{\mathrm{fp}} \cong \mathsf{P}$, category of compact polyhedra.

There is no known useful lifting of Baker-Beynon duality to the whole of $\mathrm{Alg}\,T$.

Baker-Beynon Duality over $\mathbb{Z}$

An affine map $\mathbb{R}^n \to \mathbb{R}$ is $\mathbb{Z}$-*affine* if it descends to $\mathbb{Z}^n \to \mathbb{Z}$.

A PL-map is a $\mathbb{Z}$-*map* if its affine pieces are $\mathbb{Z}$-affine.

The theory of $\mathbb{Z}$-maps between unit cubes $[0,1]^n$ is the theory of MV-algebras.

This lifts to a duality between rational polyhedra and $\mathbb{Z}$-maps, and the category of finitely presented MV-algebras.

VM and L. Spada, Ann. Pure Appl. Logic (2013) and Studia Logica (2012).

Measurable maps
(The algebra of integration)

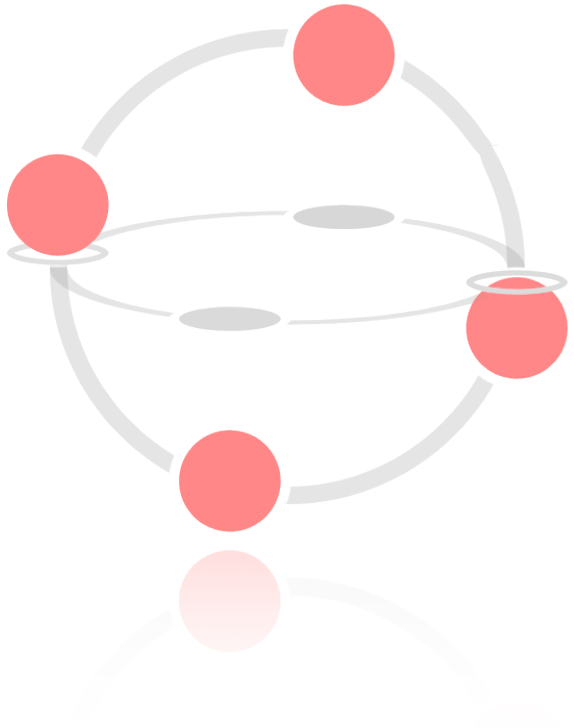
Continuous maps
δ-algebras

Arithmetic continuous maps
(Stone-Gelfand Duality for groups)

Monotone continuous maps
$\mathbf{MC}_\infty$-algebras

PL-maps
Unital vector lattices

$\mathbb{Z}$-maps
MV-algebras



Thank you for your attention.