TOPOLOGY, ALGEBRA AND CATEGORIES IN LOGIC

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# PREFACE

This volume contains the papers to be presented at TACL 2022: Topology, Algebra, and Categories in Logic to be held in Coimbra between 19th and 24th June 2022.
The volume includes the abstracts of 87 accepted contributed talks, of 9 invited talks and the programme of the conference. The abstracts are divided into invited and contributed talks and then organized in alphabetic order of the designated speaker.

The TACL conferences focus on three interconnecting mathematical themes central to the study of logic and their applications: algebraic, categorical, and topological methods.
TACL 2022 is the tenth conference in the series Topology, Algebra, and Categories in Logic (TACL, formerly TANCL). The previous editions of the series have been held in Tbilisi (2003), Barcelona (2005), Oxford (2007), Amsterdam (2009), Marseille (2011), Nashville (2013), Ischia (2015), Prague (2017) and Nice (2019).

TACL 2022 takes place at the Department of Mathematics, University of Coimbra and it is organized by the Centre for Mathematics of the University of Coimbra.

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Coimbra, June 2022

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## Invited Talks



## Contributed Talks

# Two-layered Belnapian logics for uncertainty

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Reasoning about information, its potential incompleteness, uncertainty, and contradictoriness need to be dealt with adequately. Separately, these characteristics have been taken into account by various appropriate logical formalisms and (classical) probability theory. While incompleteness and uncertainty are typically accommodated within one formalism, e.g. within various models of imprecise probability, contradictoriness and uncertainty less so — conflict or contradictoriness of information is rather chosen to be resolved than to be reasoned with. To reason with conflicting information, positive and negative support—evidence in favour and evidence against—a statement are quantified separately in the semantics. This two-dimensionality gives rise to logics interpreted over twist-product algebras or bi-lattices, the well known Belnap-Dunn logic of First Degree Entailment being a prominent example [2, 8]. Belnap-Dunn logic with its double-valuation frame semantics can in turn be taken as a base logic for defining various uncertainty measures on de Morgan algebras, e.g. Belnapian (non-standard) probabilities [11] or belief functions [15, 6].

In a spirit similar to Belnap-Dunn logic, we can introduce many-valued logics suitable to reason about such uncertainty measures. They are interpreted over twist-product algebras based on the $[0, 1]$ real interval as their standard semantics and can be seen to account for the two-dimensionality of positive and negative component of (the degree of) belief or likelihood based on potentially contradictory information, quantified by an uncertainty measure. The logics presented in this talk include expansions of Łukasiewicz or Gödel logic with a de-Morgan negation which swaps between the positive and negative semantical component. The expansions of Gödel logic, which can be equipped with a natural double-valuation frame semantics, relate to the extensions of Nelson's paraconsistent logic $N4$ [12, 13], or Wansing's paraconsistent logic $I_4C_4$ [14], with the prelinearity axiom. The resulting logics inherit both (finite) standard completeness properties, and decidability and complexity properties of Łukasiewicz or Gödel logic respectively, and allow for an efficient reasoning using the constraint tableaux calculi formalism [3].

Two-layered logics for reasoning under uncertainty of classical events were introduced in [9, 10], and developed further within an abstract algebraic framework by [7] and [1]. They separate two layers of reasoning: the inner layer consists of a logic chosen to reason about events or evidence, the connecting modalities are interpreted by a chosen uncertainty measure on propositions of the inner layer, typically a probability or a belief function, and the outer layer consists of a logical framework to reason about probabilities or beliefs. The modalities apply to inner level formulas only, to produce outer level atomic formulas, and they do not nest. Logics introduced in [9] use classical propositional logic on the inner layer, and reasoning with linear inequalities on the outer layer. [10] on the other hand use Łukasiewicz logic on the outer layer, to capture the quantitative reasoning about probabilities within a propositional logical language.

Our main objective is to utilise the apparatus of two-layered modal logics for the formalisation of reasoning with uncertain information, which itself might be non-classical, i.e., incomplete or contradictory. Many-valued logics with a two-dimensional semantics mentioned above are

*This talk is rooted in joint work with S. Frittella, D. Kozhemiachenko, O. Majer and S. Nazari.

used on the outer layer to reason about belief, likelihood or certainty based on potentially incomplete or contradictory evidence, building on Belnap-Dunn logic of First Degree Entailment as an inner logic of the underlying evidence. This results in two-layered logics suitable for various scenarios: expansions of Łukasiewicz logic are adequate in cases when aggregated evidence yields a Belnapian probability measure [4] or a belief function (on a De Morgan algebra) [6], while expansions of Gödel logic are useful to reason about comparative uncertainty in cases where it is not so, or to capture reasoning about qualitative probability [5].

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# The topological $\mu$-calculus

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Modal logic enjoys well-known topological semantics dating back to Tarski and McKinsey, where the modal $\diamond$ may be interpreted as topological closure or, alternately, as the Cantor derivative [8]. These semantics readily extend to the language of the $\mu$-calculus [7]. In this presentation we will provide a general introduction to the topological $\mu$-calculus and survey the state of the art and open questions.

Since topological operators such as the closure and interior operators are already idempotent, this version of the $\mu$-calculus behaves quite differently from its relational variant. In the closure semantics, a certain polyadic operator known as the *tangled closure* [3] is expressively complete due to results of Dawar and Otto [2]. This remains true for semantics based on the Cantor derivative over spaces satisfying a regularity condition known as $T_d$. Goldblatt and Hodkinson [5] studied the $\mu$-calculus in this setting, providing completeness results for various classes including that of metric spaces. However, many of the techniques used break down when dropping the $T_d$ assumption, which among other things allows one to embed the topological $\mu$-calculus into the relational one and thus draw on known results.

In more recent work, A. Baltag, N. Bezhanishvili and the speaker [1] have shown that completeness results for arbitrary spaces may be obtained directly via *final submodel* methods [4]. The question remains whether the expressive completeness of the tangled fragments holds in the general topological setting as well. Preliminary results by Baltag et al. [1] and Q. Gougeon [6] suggest a negative answer, although the latter also proposes a compelling candidate for an expressively complete *hybrid tangle.*

Finally, one may ask which classes of spaces are $\mu$-calculus definable without being modally definable, with no examples being known for a surprisingly long time. Gougon's thesis also provides the first such examples, leading to the introduction of *imperfect spaces.* Such classes of spaces may or may not be representative of all $\mu$-calculus definable classes, but there are reasons to conjecture that they are, especially if the question of expressive completeness of tangle-like fragments is settled in the affirmative.

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# The topological behaviour category of an algebraic theory

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In computer science, algebraic theories are used to encode computational effects [2, 3]: operations of a theory encode new language primitives which may, for example, request input from, or return output to, an external source; read and write values in a store; branch probabilistically or non-deterministically; and so on.

Many computational effects involve interaction with an external environment, and an important insight of Power and Shkaravska [4] is that the environments in question can be modelled by *comodels* of one's algebraic theory. For example, a comodel of the theory of input is a state machine which provides input tokens on demand, while a comodel of the theory of store is a state machine which handles requests to read and update the values in the store.

One can also consider *topological* comodels of an algebraic theory, where the topology tells us how much of the hidden state of a comodel is revealed via *finite* interactions with a program. The goal of this talk is to explain how the topological comodels of a given theory $\mathbb{T}$ admit a particularly nice classification: they are precisely the topological $\mathbb{B}$-sets for a certain source-étale ample topological category $\mathbb{B}$, which we call the *topological behaviour category* of the theory $\mathbb{T}$. This extends results of [1] for non-topological comodels.

If time permits, we will discuss how the kinds of topological groupoid arising in the study of combinatorial $C^*$-algebras can be re-found as topological behaviour categories of computationally natural algebraic theories.

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# Noetherian Spaces, Wqos, and their Statures

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Well-quasi-orders (wqos) are an amazingly useful notion, both in mathematics and in computer science. In a seminal paper, de Jongh and Parikh [DeJP77] showed that maximal order type of any well-partial-order (wpo) is attained, and computed it in a few cases. In her now classic Habilitationsschrift, Schmidt [Sch79] computed it in the important cases of spaces of words, and of finite trees over wpos. Those pioneering works have had considerable influence in logic (ordinal analysis) and in computer science (verification), at least.

Fifteen years ago, the first author realized that there was a natural topological generalization of well-quasi-orders, *Noetherian spaces* [Gou07]. A space is Noetherian if and only if every open subset is compact, and the notion has many equivalent definitions. It so turns out that the special kind of Noetherian spaces whose topology is Alexandroff are, in a precise sense, exactly the wqos. Over the years, it has been observed that many results and constructions that are typical of wqo theory generalize to the Noetherian setting. The goal of this presentation is to explain some recent results of ours that extend the well-known theory of maximal order types to a corresponding theory of *statures* of Noetherian space [GLL22]. Explicitly, we define the stature of a Noetherian space $X$ as the ordinal rank of $X$ in the lattice $\mathcal{H}X$ of all closed subsets of $X$, ordered by inclusion. We argue that this notion of stature coincides with maximal order types in the case of wpos, following [Kří97] or [BG08] (from whom we borrowed the term "stature"), while a more naive idea for extending the notion of maximal order type fails. We also argue that many results on maximal order types of various wpo constructions transfer to Noetherian spaces (coproducts, products, spaces of finite words, of finite multisets), with the same formulae, and we obtain new formulae for statures of a variety of Noetherian constructions that do not arise from wpos (spaces of words with the so-called prefix topology, spaces with the cofinite topology, spaces of transfinite words [GLHL22], powersets).

Instead of spending too much time on the technical details, we focus on giving a gentle introduction to the required theory of Noetherian spaces, especially seen through the lens of a computer scientist working in verification, as in [Gou10]. With this view, our hope is that statures of Noetherian spaces would be the first step in understanding the complexity of verification of so-called topological well-structured transition systems, mimicking and extending the use of maximal order types done until now (see [FFSS11, SS11], for example). Importantly, some tools that have been developed in this theory, and most notably the theory of S-representations of [FG20], initially invented in order to produce effective completions of (standard, wqo-theoretic) well-structured transition systems and generalize the so-called Karp-Miller algorithm [KM67], are crucial here, as they provide us with concrete representations of elements of $\mathcal{H}X$, allowing us to compute appropriate lower and upper bounds on their ordinal ranks.

*Speaker.

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# Why didn't locale-theorists discover DeMorganization?

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As is well known, the logical principle called 'De Morgan's Law' is the analogue for toposes of the topological property of extremal disconnectedness. In 2008 my then student Olivia Caramello showed that every topos has a largest dense subtopos satisfying De Morgan's law; this immediately implies that every locale has a largest dense sublocale which is extremally disconnected, but at the time we had no purely locale-theoretic proof of that fact. In this talk I shall present a simple frame-theoretic proof of the existence of 'DeMorganization', based on a technique which I introduced in 1989 when studying fibrewise closed sublocales, and try to explain why it was not discovered earlier. I shall also present a first contribution to the study of locales which are 'DM-averse' in the sense that their DeMorganization coincides with their Booleanization, by showing that all metric spaces are DM-averse. And I shall discuss possible connections with recent work of Rick Ball and Joanne Walters-Wayland on smallest flat and $C^*$-embedded sublocales.

# From $\{0,1\}$ to $[0,1]$: A survey of duality theorems

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The category of finite powers of the two-element set $\{0,1\}$ together with all functions between them are a Lawvere theory whose category of set-based models is the variety of Boolean algebras. The Lawvere theory is then dually equivalent to the full subcategory of free finitely generated Boolean algebras, and this duality lifts to Stone Duality between Stone spaces and Boolean algebras. Next, let us order $\{0,1\}$ as $0 < 1$, its finite powers by the product order, and let us restrict morphisms to the monotone functions. The resulting Lawvere theory is obtained from the previous one by removing some operations, notably negation. It has distributive lattices as category of set-based models, it is dually equivalent to the full subcategory of free finitely generated distributive lattices, and this duality lifts to Priestley duality between Priestley spaces and distributive lattices.

Replace $\{0,1\}$ with the real unit interval $[0,1] \subseteq \mathbb{R}$, and consider the category $\mathsf{T}'$ of its Cartesian powers up to some fixed, sufficiently large infinite cardinal, with all functions between them. Then $\mathsf{T}'$ is a Lawvere-Linton theory which provides a convenient setting to discuss a host of duality theorems—some old, some new, and some (the vast majority) uninteresting. Any subcategory $\mathsf{T}$ of $\mathsf{T}'$ that includes at least all finite powers (=finite arities) and all projection functions (=variables) provides a Lawvere-Linton theory that conceptually corresponds to a choice of structure on the Cartesian powers. Because of the inclusion $\{0,1\} \hookrightarrow [0,1]$, the dualising possibilities offered by the two-element set are subsumed; Stone and Priestley duality, for instance, may each be recovered by the appropriate choice of $\mathsf{T}$. More generally, for any such $\mathsf{T}$ one can, in principle, study the associated category of models and its duality theory. This study, though, can be expected to hold interest only insofar as the implied structure on the Cartesian powers does, for instance in light of how it relates to mathematical tradition.

Starting from this perspective I will make an attempt to survey what is known about some cases of interest, arranging them into a hierarchy of theories. The best known case is possibly that of all continuous functions, that is, of Stone-Gelfand-Yosida duality. Two further cases of interest, each rooted in tradition to different degrees of depth, are piecewise-linear functions, yielding affine Baker-Beynon duality for compact polyhedra, and monotone continuous functions, yielding Nachbin's compact ordered spaces and a duality for them. The more recent results I plan to discuss are due to various teams of authors.

# General and standard modal fuzzy logics

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Fuzzy logics are those whose algebraic semantics are classes of bounded integral residuated lattices with a continuous monoidal operation. There are three main subvarieties which allow to generate any other algebra in the class though the so-called ordinal sum construction. These are the varieties of Gödel ($\mathbb{G}$), MV ($\mathbb{MV}$) and Product ($\mathbb{P}$) algebras, algebraic semantics of Gödel ($G$), Łukasiewicz (Ł) and Product ($\Pi$) logics respectively. The subdirectly irreducible members of each one of these varieties are the linearly ordered algebras (chains). On the other hand, each one of these varieties is generated by a single algebra of its corresponding class, named the standard algebra, whose universe is $[0, 1]$. Moreover, the generated variety coincides with the generated quasi-variety. Summing up, each one of the previous logics is complete with respect to the 1-assertional logic of the corresponding class of chains and that of the standard algebra.

The F.O. extensions of the previous logics behave, however, differently. The logics arising from F.O. models evaluated over all algebras in the corresponding variety do coincide with those arising from F.O. models evaluated over the corresponding chains. In the literature, these are called the *general logics*. The *standard logics* are those arising from F.O. models evaluated over the corresponding standard algebra. In the Gödel case, the general logic coincides with the standard one. However, this is not the case for the Łukasiewicz nor Product logics. That F.O. general and standard Łukasiewicz logics are different follows as a corollary from the fact that the set of theorems of the general logic is recursively enumerable (R.E), but the set of theorems of the standard logic is not [4]. The same thing can be proven for the product case.

Modal fuzzy logics can be understood as the restriction of the previous F.O. logics to the fragment resulting from the usual translation of modal operators (and formulas in variables $\mathcal{V}$) to the formulas in the predicates language $\{R/2\} \cup \{P/1 \colon P \in \mathcal{V}\}$ as is done in the classical case. This approach yields the so-called valued Kripke models, which are Kripke models where the accessibility relation and the variables at each world are evaluated over an algebra like the ones above. The modal logics resulting from these semantics are the so-called *modal fuzzy logics*. It is relevant to note that, over the same class of models, two modal logics (i.e., consequence relations) are defined: the local and the global one. The latter is defined analogously to the F.O. entailment over arbitrary formulas (closing both premises and conclusions under universal quantifiers), while the former one refers to the notion of truth-entailment under each assignment into the model. Nevertheless, their sets of theorems coincide.

Analogously to the F.O. case, we can refer to the *general* or *standard modal fuzzy logics* whenever the evaluation is considered over all algebras (or equivalently, chains) of the corresponding variety, or only over the standard one. Furthermore, the particular cases when the accessibility relation in the Kripke model is taken as a classical binary relation (crisp) are also of special interest, since the underlying Kripke frames are classical.

In this talk, we will compare the previous general and standard logics. While in the Gödel case, the F.O. behavior immediately implies that, in all cases, the general and standard modal Gödel logics coincide, we will see how for the other two logics, the results are more varied. The global modal logics behave as the F.O. ones, namely, the general and standard logics differ. For the crisp-accessibility cases, a reasoning similar to the one from F.O. can be done. Indeed, we know that global modal standard Łukasiewicz and product logics with crisp accessibility relation are not R.E. [5]. However, the general F.O. Łukasiewicz and Product logics are axiomatizable. Henceforth, the corresponding general modal logics are R.E., implying that the standard and general logics do not coincide. On the other hand, the computational classification of the global logics with valued accessibility is not known. Nevertheless, two examples can be built to prove that also these logics differ, exploiting peculiarities of a model over the Chang algebra for the Łukasiewicz logic and of models over the analogous product algebra for the Product logic.

For what concerns local modal logics, however, we will see that the general and standard logics coincide, both for the crisp accessibility and for the valued one. This implies that the theorems of these logics (which are the same as the ones from the global logics) coincide too. In the Łukasiewicz cases, this equality can be proven relying in the F.O. completeness with respect to witnessed models (those in which, for each quantified formula, there is an assignment in the model where the formula without the quantifier takes the same value as the quantified one), both for arbitrary models and also for standard ones [1]. For the Product logic, we can prove the claim for the models with valued accessibility relation by relying in the details of a proof of decidability of the Description Logic over the standard product algebra [3]. This does not serve to tackle the case with crisp accessibility, which can nevertheless be proven by a different approach using the completeness of F.O. product logic with respect to models evaluated over a certain algebra (the one arising via Cignoli-Torrens functor from the lexicographic sum $\mathbb{R}^Q$) [2]. Using models valued over this algebra we identify certain conditions, that can be expressed with finitely many propositional formulas, and that capture all relevant information about the modal operations. In this fashion, the modal general product logic can be faithfully encoded within the the propositional one, and so, it is possible to rely in the standard completeness of the latter to prove that the general and standard modal logics also coincide.

Furthermore, this latter proof also will allow us to answer positively to the open question of the decidability of (local) crisp-accessibility modal standard product logic.

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# Spectra and subspectra arising from $\ell$-groups and commutative rings

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The spectrum of an Abelian $\ell$-group is defined as the set of all its prime $\ell$-ideals endowed with the hull-kernel topology. The real spectrum of a commutative unital ring is an ordered analogue of its Zariski spectrum. We give a complete list of the containments and non-containments between the classes of $\ell$-spectra and real spectra, and their spectral subspaces, for $\ell$-groups and rings, highlighting the differences between the cases of structures of cardinality either countable, $\aleph_1$, or $\aleph_2$. We also give a hint of the methods used: semilinear algebra / real algebraic geometry (cases $\aleph_0$, $\aleph_1$), category theory, infinite combinatorics, and logic (beyond $\aleph_1$, especially $\aleph_2$).

# Dependence logic and team semantics

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Dependence logic was introduced by Väänänen (2007) as a logical formalism for reasoning about dependence and independence relations. The logic adds to first-order logic a new type of atomic formulas, called dependence atoms, to specify explicitly the dependency between variables. Dependence logic adopts the team semantics of Hodges (1997). The basic idea of team semantics is that dependency properties can only manifest themselves in multitudes, and thus formulas of dependence logic are evaluated on a model with respect to sets of assignments (called teams) instead of single assignments (as in the usual Tarskian semantics).

In this talk, we survey some basic results for first-order and propositional dependence logic. We also discuss approaches to generalize team semantics and define propositional dependence logics based on intuitionistic and intermediate logics.

# An approach à la de Vries to compact Hausdorff spaces and closed relations

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De Vries [3] obtained a duality for the category $\mathsf{KHaus}$ of compact Hausdorff spaces and continuous functions. The objects of the dual category $\mathsf{DeV}$ are complete boolean algebras equipped with a proximity relation, known as de Vries algebras, and the morphisms are functions satisfying certain conditions. One drawback of $\mathsf{DeV}$ is that composition of morphisms is not usual function composition. We propose an alternative approach, where morphisms between de Vries algebras are certain relations and composition of morphisms is usual relation composition.

For our purpose, it is more natural to start with the category $\mathsf{KHaus}^\mathsf{R}$ whose objects are compact Hausdorff spaces and whose morphisms are closed relations (i.e., relations $R\colon X \to Y$ where $R$ is a closed subset of $X \times Y$). This category was studied in [2], and earlier in [6] in the more general setting of stably compact spaces. The latter paper establishes a duality for $\mathsf{KHaus}^\mathsf{R}$ that generalizes Isbell duality [5] between $\mathsf{KHaus}$ and the category of compact regular frames and frame homomorphisms. This is obtained by generalizing the notion of a frame homomorphism to that of a preframe homomorphism. However, a similar duality in the language of de Vries algebras remained problematic (see [2, Rem. 3.14]). We resolve this problem as follows.

As in de Vries duality, with each compact Hausdorff space $X$ we associate the de Vries algebra $(\mathsf{RO}(X), \prec)$, where $\mathsf{RO}(X)$ is the complete boolean algebra of regular open subsets of $X$ and $\prec$ is defined on $\mathsf{RO}(X)$ by $U \prec V$ iff $\mathsf{cl}(U) \subseteq V$. The key is to associate with each closed relation $R\colon X \to Y$ the relation $S_R\colon \mathsf{RO}(X) \to \mathsf{RO}(Y)$ given by

$$U \, S_R \, V \text{ iff } R[\mathsf{cl}(U)] \subseteq V$$

($R[-]$ denotes the direct image under $R$). This defines a covariant functor from $\mathsf{KHaus}^\mathsf{R}$ to the category $\mathsf{DeV}^\mathsf{S}$ of de Vries algebras and compatible subordination relations between them (i.e., subordination relations $S\colon A \to B$ satisfying $S \circ \prec_A = S = \prec_B \circ S$). Our main result states that this functor is an equivalence. We then prove that this equivalence further restricts to an equivalence between $\mathsf{KHaus}$ and the wide subcategory $\mathsf{DeV}^\mathsf{F}$ of $\mathsf{DeV}^\mathsf{S}$ whose morphisms satisfy additional conditions. This yields an alternative to de Vries duality. The main advantage of $\mathsf{DeV}^\mathsf{F}$ is that composition of morphisms is usual relation composition.

While our main result establishes that $\mathsf{KHaus}^\mathsf{R}$ is equivalent to $\mathsf{DeV}^\mathsf{S}$, the choice of direction of morphisms is ultimately a matter of taste since morphisms are relations. In fact, both $\mathsf{KHaus}^\mathsf{R}$ and $\mathsf{DeV}^\mathsf{S}$ are dagger (and thus self-dual) categories, and hence our results could alternatively be stated in the language of duality rather than equivalence. (To obtain a duality, one associates to a closed relation $R\colon X \to Y$ the compatible subordination relation $T_R\colon \mathsf{RO}(Y) \to \mathsf{RO}(X)$ defined by $V \; T_R \; U$ iff $R^{-1}[\mathsf{cl}(V)] \subseteq U$. This assignment exhibits a duality between $\mathsf{KHaus}^\mathsf{R}$ and $\mathsf{DeV}^\mathsf{S}$, which restricts to a duality between $\mathsf{KHaus}$ and a wide subcategory of $\mathsf{DeV}^\mathsf{S}$ whose morphisms satisfy conditions that are "dual" to those satisfied by the morphisms of $\mathsf{DeV}^\mathsf{F}$.)

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Our proof builds on a generalization of Halmos duality, which in turn generalizes Stone duality. By Stone duality, the category of Stone spaces (i.e. zero-dimensional compact Hausdorff spaces) and continuous maps is dually equivalent to the category of boolean algebras and boolean homomorphisms. Halmos [4] generalized this result to a duality between the category of Stone spaces and continuous relations and the category of boolean algebras and functions preserving finite joins. This approach can be further generalized by working with closed relations instead of continuous ones. As shown in [1], a closed binary relation on a Stone space $X$ corresponds to a subordination relation on the boolean algebra $\mathsf{Clop}(X)$ of clopen subsets of $X$. The notion of a subordination relation on a boolean algebra generalizes to that of a subordination relation between two boolean algebras. This yields the category $\mathsf{BA}^\mathsf{S}$ of boolean algebras and subordination relations between them (identity is $\leq$ and composition is relation composition). There is an equivalence (and also a dual equivalence) of categories between $\mathsf{BA}^\mathsf{S}$ and the full subcategory $\mathsf{Stone}^\mathsf{R}$ of $\mathsf{KHaus}^\mathsf{R}$ consisting of Stone spaces. This result generalizes Stone and Halmos dualities. For a more general result in the context of Priestley spaces and bounded distributive lattices we refer to [7]. Using Karoubi envelopes and Gleason covers, we derive our equivalence between $\mathsf{KHaus}^\mathsf{R}$ and $\mathsf{DeV}^\mathsf{S}$ from the equivalence between $\mathsf{Stone}^\mathsf{R}$ and $\mathsf{BA}^\mathsf{S}$.

In [8, Thm. 26], the category of stably compact spaces and continuous functions was shown to be equivalent to the category of strong proximity lattices and approximable relations (see also [10] for a proof that explicitly uses Karoubi envelopes). This equivalence was specialized to compact Hausdorff spaces in [9]. In this case, to a compact Hausdorff space $X$ is associated the set $\{(U, K) \mid U \text{ open subset of } X,\ K \text{ closed subset of } X,\ U \subseteq K\}$, equipped with an appropriate structure. The difference between this assignment and ours (based on regular open sets) makes the axioms of the structures in [9] incomparable to ours.

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# Cohomological refinements of $k$-consistency and $k$-equivalence

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The seminal work of Kochen and Specker [11] showed that quantum mechanics is fundamentally *contextual*: the properties of a quantum system must be considered relative to the context in which they are measured. There is no consistent way of assigning values to all the observables. In [3, 4, 2], contextuality was studied from a sheaf-theoretic point of view, and sheaf cohomology was used to characterise the obstructions to having a consistent global assignment to all the variables. One could say that cohomology detects the *holes* which prevent there being a consistent picture of a global *whole*.

Constraint satisfaction is an important algorithmic paradigm which allows the application of structural methods to central questions of complexity theory. The "non-uniform" version CSP($B$) for a fixed finite $\sigma$-structure $B$, where $\sigma$ is a finite relational vocabulary, asks for an instance given by a finite $\sigma$-structure $A$ whether there is a $\sigma$-homomorphism $A \to B$. The celebrated Feder-Vardi Dichotomy Conjecture [8] asked whether for every $B$, CSP($B$) is either polynomial-time solvable, or NP-complete. This conjecture was recently proved by Bulatov and Zhuk [5, 12].

Recently, Adam Ó Conghaile has pointed out surprisingly close connections between these two, prima facie completely unrelated topics [7], further developed in [1].

- The idea of $k$-consistency in constraint satisfaction, an approximation method which yields exact results in a wide range of cases, is naturally represented as the coflasquification (dual to the well-known Godement construction [9]) of a sheaf of partial homomorphisms.
- These representations take the same form as the sheaf-theoretic representations of contextuality in [3]. This in turn allows the cohomological criteria for contextuality introduced in [4, 2] to be used to give a computationally efficient refinement of $k$-consistency.
- The results in [4, 2] can be leveraged to show that this refined version of $k$-consistency gives exact results for all affine templates, which form one of the main tractable classes for which the standard $k$-consistency algorithm fails.
- Current work is aimed at determining the exact power of the cohomological refinement of $k$-consistency.
- The same ideas can be adapted to give a very similar analysis for the widely studied Weisfeiler-Leman equivalences [10], which give polynomial-time approximations to graph and structure isomorphism. Cohomological refinements of these equivalences can then be introduced, and are shown in [7] to defeat various families of counter-examples based on the Cai-Furer-Immerman construction [6], which is paradigmatic in finite model theory.

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# Hausdorff Polynomial Functors

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We study endofunctors on categories of metric spaces analogous to the Kripke polynomial set functors. The latter form a 'well-behaved' class of functors $F$ including a number of examples where $F$-coalgebras are well-known state-based systems. (For example Kripke structures.) Kripke polynomial functors are the set functors $F$ obtained from Id, the finite power-set functor $\mathcal{P}_f$ and the constant functors $A$ ($A$ any set) by using products, coproducts, and composition:

$$F ::= \mathcal{P}_f \,|\, \mathrm{Id} \,|A|\prod_{i\in I} F_i \mid \coprod_{i\in I} F_i | FF$$

(Various other versions are used in the literature, restricting e.g. coproducts to finite ones etc.)

It follows from the results of Worrell [7] that each such functor $F$ has a terminal coalgebra obtained in $\omega+\omega$ steps of the construction of terminal coalgebras introduced (in dual form) in [1]. This transfinite construction is given by $V_0 = 1$, the terminal object, $V_{i+1} = FV_i$ for every $i < \omega+\omega$, and $V_\omega$ is the limit of the $\omega^{op}$-chain $V_i$ ($i < \omega$). For every Kripke polynomial functor $F$ the terminal coalgebra is the limit $\lim\limits_{i<\omega+\omega} V_i$. Consequently, every Kripke polynomial functor $F$ generates a cofree comonad $\overline{F}$ obtained as a limit of the chain $\overline{V}_i$ ($i < \omega+\omega$) of endofunctors defined by $\overline{V}_0 = \mathrm{Id}$, $\overline{V}_{i+1} = F\cdot\overline{V}_i + \mathrm{Id}$ and $\overline{V}_\omega = \lim\limits_{i<\omega}\overline{V}_i$: we have $\overline{F} = \lim\limits_{i<\omega+\omega}\overline{V}_i$.

Recall that $\mathcal{P}_f$ is the monad on **Set** of semilattices with zero.

For coalgebraic modal logic the Vietoris endofunctor on the category of Stone spaces (assigning to a space all compact subsets with the Vietoris topology) plays an important role. This led Kurz et al. [5] to introduce Vietoris polynomial functors by the analogous grammar to above: just $\mathcal{P}_f$ is substituted by the Vietoris functor.

We now consider similar collections of endofunctors on the category **Met** of extended metric spaces (that is, distance $\infty$ is admitted) and nonexpanding maps. And on its full subcategories **CMet** of complete spaces and **KMet** of compact spaces. The role of $\mathcal{P}_f$ is played be the *Hausdorff functor* assingning to a space $X$ the Hausdorff space $\mathcal{H}X$ of all nonempty compact subsets with the Hausdorff metric

$$d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$$

where $d(a,B) = \inf_{b\in B} d(a,b)$. The functor $\mathcal{H}$ preserves completeness, and it thus yields an endofunctor of **CMet**, also denoted by $\mathcal{H}$. This is the semilattice monad on **CMet**, as proved in [3]. (Surprisingly, the functor $\mathcal{H}$ is finitary on **CMet**, see [2].)

Analogously, the semilattice monad on **Met** assigns to $X$ the space $\mathcal{H}_f X$ of all nonempty finite subsets with the Hausdorff metric, as proved in [6].

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**Definition.** *Hausdorff polynomial functors* on **CMet** are the endofunctors $F$ obtained from the Hausdorff functor, Id and the constant functors $A$ (any object) by using products, coproducts, and composition:

$$F ::= \mathcal{H} | \operatorname{Id} | A | \prod_{i \in I} F_i \mid \coprod_{i \in I} F_i | FF$$

We now obtain a result about cofree comonads for the above functors analogous to Kripke polynomial functors. An *isometric embedding* is a morphism of **Met** preserving distances. A cone $f_i \colon A \to A_i$ $(i \in I)$ is called *isometric* if $\langle f_i \rangle \colon A \to \prod_{i \in I} A_i$ is an isometric embedding.

**Theorem.** *The Hausdorff functor $\mathcal{H}$ on* **CMet** *preserves isometric embeddings and their wide intersections. Moreover, it preserves isometric cones of $\omega^{\mathrm{op}}$-chains.*

**Corollary.** Every Hausdorff polynomial functors $F$ on **CMet**
(1) has a terminal coalgebra obtained in $\omega + \omega$ steps: $\nu F = \lim_{i < \omega + \omega} V_i$, and
(2) generates a cofree comonad in $\omega + \omega$ steps: $\overline{F} = \lim_{i < \omega + \omega} \overline{V}_i$.

It follows that the category of coalgebras for $F$ is complete.

An analogous corollary holds for **Met**, where the polynomial functors are as above with $\mathcal{H}_f$ replacing $\mathcal{H}$. (Actually, that corollary also holds for **Met** *without* this replacement: every endofunctor on **Met** as in the above definition has a terminal coalgebra obtained in $\omega + \omega$ steps.)

The situation with **KMet** is completely different: as demonstrated by Hofmann and Nora [4], the Hausdorff polynomial functor $\mathcal{H} + 1$ on this subcategory does not have a terminal coalgebra. In op.cit. the name Hausdorff polynomial functors is used in a different manner: the elements of $\mathcal{H}X$ are the closed subsets of $X$ rather than the compact ones. (On the category **KMet** this makes no difference, of course.) Terminal coalgebras for such endofunctors on **CMet** are proved not to exist in op.cit.

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# Projectivity in (bounded) commutative integral residuated lattices

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We approach the study of projective algebras in varieties of (bounded) commutative integral residuated lattices. Our point of view is going to be algebraic, however the reader may keep in mind that projectivity is a categorical concept, and therefore our findings pertain to the corresponding algebraic categories as well. Being projective in a variety of algebras, or in any class containing all of its free objects, corresponds to being a retract of a free algebra, and projective algebras contain relevant information both on their variety and on its lattice of subvarieties.

In particular, as first noticed by McKenzie [8], there is a close connection between projective algebras in a variety and *splitting algebras* in its lattice of subvarieties. The notion of splitting algebra comes from lattice theory, and studying splitting algebras is particularly useful to understand lattices of subvarieties, since a splitting algebra divides a subvariety lattice in a disjoint union of a principal filter defined by its generated variety, and a principal ideal. The varieties of algebras that are the object of our study are relevant both in the realm of algebraic logic and from a purely algebraic point of view. In fact, residuated structures arise naturally in the study of many interesting algebraic systems, such as ideals of rings or lattice-ordered groups, besides encompassing the equivalent algebraic semantics (in the sense of [2])of substructural logics. We refer the reader to [6] for detailed information on this topic. The Blok-Pigozzi notion of algebraizability entails that the logical deducibility relation is fully and faithfully represented by the algebraic equational consequence of the corresponding algebraic semantics, and therefore logical properties can be studied algebraically, and viceversa. Substructural logics are a large framework and include most of the interesting non-classical logics: intuitionistic logics, relevance logics, and fuzzy logics to name a few, besides including classical logic as a special case. Therefore, substructural logics on one side, and residuated lattices on the other, constitute a wide unifying framework in which very different structures can be studied uniformly.

The investigation of projective structures in particular varieties of residuated lattices has been approached by several authors (see for instance [1],[3], [4], [5], [7]). However, to the best of our knowledge, no effort has yet been done to provide a uniform approach in a wider framework, which is what we attempt to start here. Besides some general findings on $\mathsf{FL}_e w$-algebras, our main results will concern varieties with particular properties: varieties closed with respect to the ordinal sum construction and varieties where the lattice order is actually the inverse divisibility ordering. With these methods, we show that several interesting varieties in the realm of algebraic logic have the property that every finitely presented algebra is projective, among which: locally finite varieties of hoops, cancellative hoops, product algebras. As a consequence of some general results about projectivity in varieties closed under ordinal sums, we show an alternative proof of the characterization of finite projective Heyting algebras. In the

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more general setting, via the connection with splitting algebras, we show that the only finite projective algebra in $\mathsf{FL}_e w$ is the two elements Boolean algebra **2**, while we identify a large class of structures where all finite Boolean algebras are projective. Interestingly, the study of projective algebras in this realm has a relevant logical application.

Indeed, following the work of Ghilardi [7], the study of projective algebras in a variety is strictly related to unification problems for the corresponding logic. More precisely, solving a unification problem is equivalent to finding a homomorphism from a suitable finitely presented algebra $\mathbf{A}$ in a projective algebra and if this is possible then $\mathbf{A}$ is said to be **unifiable**. So the case in which a finitely presented algebra is unifiable if and only if it is projective (as is the case in the examples quoted above) is noteworthy in unification theory. We will illustrate some immediate consequences of this connection.

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# Algebraizability as an algebraic structure

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By *logic* we mean a pair $(\Sigma, \vdash)$ where $\Sigma$ is a signature, i.e. a collection of connectives with finite arities, and $\vdash$ is an idempotent, increasing, monotone, finitary and structural consequence relation on the set $Fm_\Sigma(X)$ of formulas over $\Sigma$ with a set $X$ of variables. A *translation* between logics $(\Sigma, \vdash) \to (\Sigma', \Vdash)$ is a map $f\colon Fm_\Sigma(X) \to Fm_{\Sigma'}(X)$ induced by an arity preserving map $\Sigma \to Fm_{\Sigma'}(X)$. It is called *conservative* if $\gamma \vdash \varphi \;\Leftrightarrow\; f(\Gamma) \vdash' f(\varphi)$.

**Remote Algebraizability**

A *remote algebraization* of a logic $L$ is a jointly conservative family of translations $f_i\colon L = (\Sigma, \vdash) \to (\Sigma_i, \vdash_i) = L_i$ to algebraizable logics $L_i$. Remote algebraization has been introduced by Bueno et al. in [**BCC**] and successfully applied to non-algebraizable, and generally badly behaved, paraconsistent logics.

Recall that a logic is algebraizable if it has a set $\Delta$ of equivalence formulas and a set $\langle \delta, \epsilon \rangle$ of pairs of formulas satisfying certain syntactic conditions given in [**BP**, Thm. 4.7].

**Definition 1.** *A logic is called $(n, m)$-algebraizable, if it admits an algebraizing pair $(\Delta, \langle \delta, \epsilon \rangle)$ for which $\Delta$ consists of at most $n$ formulas and $\langle \delta, \epsilon \rangle$ consists of at most $m$ pairs of formulas.*

The following construction forces a logic to become $(n, m)$-algebraizable:

**Definition 2.** *Given a logic $L = (\Sigma, \vdash)$, one defines the logic $L \otimes A_{n,m} = (\Sigma', \vdash')$ as follows:*

*$\Sigma'$ is obtained by adjoining binary connectives $\Delta_1, \ldots, \Delta_n$ and unary connectives $\delta_1, \ldots, \delta_m$, $\epsilon_1, \ldots, \epsilon_m$ to the signature $\Sigma$. We abbreviate $\Delta = \{\Delta_1, \ldots, \Delta_n\}$ and $\langle \delta, \epsilon \rangle = \{\langle \delta_1, \epsilon_1 \rangle, \ldots, \langle \delta_m, \epsilon_m \rangle\}$.*

*$\vdash'$ is the consequence relation generated by the rules of $\vdash$ and the rules making $(\Delta, \langle \delta, \epsilon \rangle)$ into an algebraizing pair.*

Clearly we have an inclusion $L \to L \otimes A_{n,m}$ which is a translation, and this is a generic candidate for a remote algebraization.

**Proposition 3.** *A logic $L$ admits a remote algebraization by a finite family of translations if and only if the translation $L \to L \otimes A_{n,m}$ is conservative for some $n, m \in \mathbb{N}$.*

Using this equivalence, we can characterize the finitely remotely algebraizable logics:

**Theorem 4.** *A logic $L = (\Sigma, \vdash)$ is remotely algebraizable by a finite family of translations if and only if one of the following conditions holds:*

*(1) $L$ has theorems.*

*(2) $L$ admits no derivation of the form $\{x\} \vdash \varphi$ in which the variable $x$ does not occur in $\varphi$.*

The theorem and its proof also elucidate what are the possible obstructions to the algebraizability of a logic: On the one hand it can be missing connectives for forming an algebraizing pair – this is what we try to remedy with the construction of Def. 2. On the other hand it can be a kind of explosive behaviour, excluded by condition (2), which even prevents adding such connectives in a conservative manner!

**Algebraizability as algebraic structure**

We consider the category **HoLog** whose objects are logics and whose morphisms are equivalence classes of translations, where translations $f, g$ are equivalent iff $f(\varphi) \dashv\vdash g(\varphi)$ for all $\varphi$ in the domain.

The construction $L \mapsto L \otimes A_{n,m}$ of Def. 2 is part of a functor $\mathfrak{A}_{n,m}\colon \mathbf{HoLog} \to \mathbf{HoLog}$. We have natural transformations $\mathrm{id} \to \mathfrak{A}_{n,m}$ given by the inclusions of Prop. 3 and $\mathfrak{A}_{n,m} \circ \mathfrak{A}_{n,m} \to \mathfrak{A}_{n,m}$ given by identifying the two copies of formulas of the algebraizing pair.

**Theorem 5.** *(1) The functor $\mathfrak{A}_{n,m}$ with these two natural transformations is a finitary monad on* **HoLog**. *(2) A logic is $(n,m)$-algebraizable if and only if it admits an algebra structure for the monad $\mathfrak{A}_{n,m}$ (3) A logic admits at most one $\mathfrak{A}_{n,m}$-algebra structure. (4) The category of $\mathfrak{A}_{n,m}$-algebras is equivalent to the category of $(n,m)$-algebraizable logics and morphisms that preserve algebraizing pairs.*

From previous results of the authors one can derive that **HoLog** is locally finitely presentable. Results on monads and accessible categories then yield the following consequences:

**Theorem 6.** *(1) The category of $(n,m)$-algebraizable logics, and equivalence classes of algebraizing pair preserving translations is locally finitely presentable.*

*(2) The category* **HoAlg** *of algebraizable logics, and equivalence classes of algebraizing pair preserving translations is accessible.*

In particular the categories of $(n,m)$-algebraizable logics are equivalent to categories of models of finite limit theories, and the category **HoAlg** is equivalent to a category of models of an infinitary first order theory. This is a priori not at all clear, given the several places in which the definitions of logics and algebraizable logics refer to subsets.

**Other Leibniz classes**

The setup of a filtered collection of logics like the $(n,m)$-algebraizable logics above is precisely mirrored in Jansana's and Moraschini's definition of Leibniz class [**JaMo**]. In the final part of the talk we discuss how much of the above results extend to general Leibniz classes. For example for protoalgebraic logics, everything up to Thm. 5(1) and (2) goes through, but since the implication formulas witnessing protoalgebraicity are not unique, as an analog of Thm. 5(4) we obtain we obtain an equivalence with the category of protoalgebraic logics *with a chosen set of witnessing formulas.*

The analog of the construction of Def. 2 is actually a *coproduct* with a generic protoalgebraic logic, and this allows for a descent theory by which one can detect whether a logic is protoalgebraic to begin with. This is in contrast with algebraizable logics, where the construction is not a coproduct and where no such detection mechanism exists.

We finish by sketching an emerging general theory of monads and descent for Leibniz classes.

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# Generalized subspaces in the duality of $T_D$-spaces

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A space $X$ is said to be $T_D$ if every point $x \in X$ has an open neighborhood $U$ such that $U - \{x\}$ is open (cf. [3]). This is a weak separation axiom, stronger than $T_0$ and weaker than $T_1$, and it plays an important role in point-free topology (see, for instance, [6]).

Actually, it can be argued that the importance of the $T_D$ axiom is similar to that of sobriety because both concepts are, in a certain sense, dual to each other [4] — see for example the following two symmetric characterizations:

- A space $X$ is sober if and only if there is no proper subspace inclusion $\iota\colon X \hookrightarrow Y$ such that the associated frame homomorphism $\Omega(\iota)$ is an isomorphism.
- A space $X$ is $T_D$ if and only if there is no proper subspace inclusion $\iota\colon Y \hookrightarrow X$ such that the associated frame homomorphism $\Omega(\iota)$ is an isomorphism.

Now, the classical adjunction

$$\begin{tikzcd} \mathsf{Top} \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathsf{Loc} \arrow[l, bend left, "\Sigma"] \end{tikzcd}$$

between topological spaces and locales restricts to an equivalence between sober spaces and spatial locales; and it was shown by Banaschewski and Pultr in [4] that there is a similar situation for the $T_D$-case. More precisely, there is an adjunction

$$\begin{tikzcd} \mathsf{Top}_D \arrow[r, bend left, "\Omega"] \arrow[r, phantom, "\perp"] & \mathsf{Loc}_D \arrow[l, bend left, "\Sigma'"] \end{tikzcd}$$

where $\mathsf{Top}_D$ denotes the category of $T_D$-spaces and their continuous maps, and $\mathsf{Loc}_D$ is a certain non-full subcategory of $\mathsf{Loc}$. This adjunction restricts to an equivalence between $\mathsf{Top}_D$ and the subcategory of $\mathsf{Loc}_D$ consisting of $T_D$-spatial locales. Since $\Omega$ is full and faithful, one may regard $\mathsf{Loc}_D$ as a category of generalized $T_D$-spaces.

In this talk, following [1, 2], we shall discuss the basic properties of the category $\mathsf{Loc}_D$, paying special attention to its regular subobject lattices (i.e., the lattices of generalized subspaces in the $T_D$-duality).

We will provide $T_D$-analogues of some well-known constructions in the theory of locales (e.g., the assembly of a frame), and explore some of their applications in point-free topology, especially in connection with $T_D$-spatiality. We will also stress the similarities and differences between the classical sober-spatial duality and the $T_D$-duality (e.g., the functorial behaviour of the assembly).

Parts of this talk are joint work with Javier Gutiérrez García and Anna Laura Suarez.

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# Contextuality in logical form: Duality for transitive partial CABAs

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Partial Boolean algebras were introduced by Kochen and Specker in their seminal work on contextuality in quantum mechanics [3, 2], as a natural (algebraic-)logical setting for contextual systems, corresponding to a calculus of partial propositional functions. They provide an alternative to traditional Birkhoff–von Neumann quantum logic [1] in which operations such as conjunction and disjunction are partial, being only defined in the domain where they are physically meaningful. In the key example of the projectors on a Hilbert space, the operations are only defined for commuting projectors, which correspond to properties of the quantum system that can be tested simultaneously.

We extend the classical Lindenbaum–Tarski dualities between finite sets and finite Boolean algebras, and more generally between sets and complete atomic Boolean algebras (CABAs), to the setting of (transitive) partial Boolean algebras. Specifically, we establish a dual equivalence between the category of transitive partial CABAs and a category of exclusivity graphs with an appropriate notion of morphism.

The vertices of an exclusivity graph may be interpreted as *possible worlds* of maximal information, with edges representing logical incompatibility or mutual exclusivity between two worlds. The classical case corresponds to complete graphs, as all possible worlds are mutually exclusive. Similarly, the appropriate notion of morphism is relaxed from functions to certain kinds of relations. From an exclusivity graph, a transitive partial CABA is constructed whose elements are sets of mutually exclusive worlds (cliques of the graph) modulo an equivalence relation. This equivalence identifies cliques that jointly exclude the same set of worlds, i.e. that have the same neighbourhood. The main result shows, in particular, that any transitive partial CABA can be recovered in this fashion from its graph of atoms with the logical exclusivity relation.

We also give an explicit construction of the free transitive partial CABA on a set of propositions with a compatibility relation, via an adjunction between compatibility graphs and exclusivity graphs that generalises the powerset self-adjunction from the classical case.

The duality reveals a connection between the algebraic-logical setting of partial Boolean algebra and the graph-theoretic approach to contextuality of Cabello–Severini–Winter. Under it, a transitive partial CABA witnessing contextuality, in the Kochen–Specker sense that it has no homomorphism to the two-element Boolean algebra, corresponds to a graph with no 'points', i.e. with no maps from the singleton graph, which are in bijection with stable, maximum clique transversal sets.

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[*] Speaker.

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# Farness via Galois adjunctions and a separation theorem for uniform frames

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In classical topology, the relation of farness has been studied in [6] and used to prove an insertion result for uniform spaces in [4]. We will present the pointfree version of this notion. Instead of aproaching it in a geometrical way, we choose to describe it algebraically, in terms of Galois adjunctions. As an application, we characterize uniform frame homomorphisms and give a separation result for uniform frames.

A subset $U \subseteq L$ of a frame $L$ is a *cover* if $\bigvee U = 1$. For each cover $U$ of $L$ let $\mathsf{S}_U$ be the *star operator*, that is

$$(x \mapsto Ux = \bigvee\{u \in U \mid u \wedge x \neq 0\})\colon L \to L$$

and let $\mathsf{P}$ be the *pseudocomplement operator*

$$(x \mapsto x^* = \bigvee\{y \in L \mid y \wedge x = 0\})\colon L \to L.$$

Notice that $\mathsf{P}$ does not depend on the cover $U$. While $\mathsf{P}$ is a self-adjoint Galois map (i.e. the pair $(\mathsf{P}, \mathsf{P})$ is a dual Galois adjunction), the star operator $\mathsf{S}_U$ is a left adjoint Galois map with right adjoint $\widetilde{\mathsf{S}}_U$ given by $\widetilde{\mathsf{S}}_U(y) = \bigvee\{x \in L \mid Ux \leq y\}$ (i.e. the pair $(\mathsf{S}_U, \widetilde{\mathsf{S}}_U)$ is a Galois adjunction) and we have the following diagram of adjunctions

$$\begin{tikzcd} L \arrow[r, bend left, "\mathsf{S}_U"] \arrow[r, phantom, "\perp"] & L \arrow[l, bend left, "\widetilde{\mathsf{S}}_U"] \arrow[r, bend left, "\mathsf{P}"] \arrow[r, phantom, "\perp^{op}"] & L \arrow[l, bend left, "\mathsf{P}"] \end{tikzcd}$$

Denoting by $\mathsf{F}_U$ the composite $\mathsf{P}\mathsf{S}_U$ (which can be proved to be equal to $\widetilde{\mathsf{S}}_U\mathsf{P}$), elements $a, b \in L$ are $U$*-far* if $a \leq \mathsf{F}_U(b)$ (or, equivalently, $b \leq \mathsf{F}_U(a)$).

Since we are interested in farness in uniform frames, we recall some notions that were first studied in [5] (see [3] for more information). A *uniformity* on a frame $L$ is a system $\mathcal{U}$ of covers such that

(U1) $U \in \mathcal{U}$ and $U \leq V$ implies $V \in \mathcal{U}$,

(U2) $U, V \in \mathcal{U}$ implies $U \wedge V \in \mathcal{U}$,

(U3) for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $VV \leq U$, and

(U4) for every $a \in L$, $a = \bigvee\{b \mid b \lhd_{\mathcal{U}} a\}$

where $U \wedge V = \{u \wedge v \mid u \in U, v \in V\}$, $VV = \{\mathsf{S}_V(v) \mid v \in V\}$ and we write $b \lhd_{\mathcal{U}} a$ if $\mathsf{S}_U(b) \leq a$ for some $U \in \mathcal{U}$. Without (U4), we say $\mathcal{U}$ is a *preuniformity.* A frame homomorphism $f\colon L \to M$ is a *uniform homomorphism* $(L,\mathcal{U}) \to (M,\mathcal{V})$ if $h[U] \in V$ for every $U \in \mathcal{U}$.

As an example, we can consider the frame of reals $\mathfrak{L}(\mathbb{R})$ presented by generators $(p, —)$ and $(—, p)$ for all rationals $p$ and a given set of relations ([3]). This frame carries its metric uniformity (see [2]). Thus, for a frame $L$ with a (pre)uniformity $\mathcal{U}$, we say a real-valued function $f\colon \mathfrak{L}(\mathbb{R}) \to L$ is *uniformly continuous* if it is a uniform frame homomorphism with respect to the metric uniformity of $\mathfrak{L}(\mathbb{R})$ and $\mathcal{U}$. We have the following characterization for real-valued uniform frame homomorphsims:

**Theorem.** *Let $(L,\mathcal{U})$ be a (pre)uniform frame. The following are equivalent for any frame homomorphism $f\colon \mathfrak{L}(\mathbb{R}) \to L$:*

(i) *$f$ is uniformly continuous.*

(ii) *For every $\delta \in \mathbb{Q}^+$, there is some $U \in \mathcal{U}$ such that $f(—, r)$ and $f(s, —)$ are $U$-far for all $r, s \in \mathbb{Q}$ such that $s - r > \frac{1}{\delta}$.*

As an application one can obtain an Urysohn-type separation result for uniform frames, namely:

**Theorem.** *Let $\mathcal{U}$ be a (pre)uniformity on a frame $L$. If $a$ and $b$ are $U$-far, for some $U \in \mathcal{U}$, then there is a bounded uniformly continuous $f : \mathfrak{L}(\mathbb{R}) \to L$ such that $f(0, —) \leq a^*$ and $f(—, 1) \leq b^*$.*

We will present the proof of this result which features a purely algebraic (order-theoretic) construction; it mainly relies on the Galois adjunction that defines the relation of farness.

This talk is based on the preprint [1] and it is a joint work with Jorge Picado.

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# Difference–restriction algebras of partial functions: axiomatizations and representations

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**Abstract**

We investigate the representation and complete representation classes for algebras of partial functions with the signature of relative complement and domain restriction. We provide and prove the correctness of a finite equational axiomatization for the class of algebras representable by partial functions. As a corollary, the same equations axiomatize the algebras representable by injective partial functions. For complete representations, we show that a representation is meet complete if and only if it is join complete. Then we show that the class of completely representable algebras is precisely the class of atomic and representable algebras. As a corollary, the same properties axiomatize the class of algebras completely representable by injective partial functions. The universal-existential-universal axiomatization this yields for these complete representation classes is the simplest possible, in the sense that no existential-universal-existential axiomatization exists.

The study of algebras of partial functions is an active area of research that investigates collections of partial functions and their interrelationships from an algebraic perspective. In pure mathematics, algebras of partial functions arise naturally as structures such as inverse semigroups [10], pseudogroups [7], and skew lattices [8]. In theoretical computer science, they appear in the theories of finite state transducers [3], computable functions [6], deterministic propositional dynamic logics [5], and separation logic [4]. The partial functions are treated as abstract elements that may be combined algebraically using various natural operations. Many different selections of operations have been considered, each leading to a different class/category of abstract algebras (see [9, §3.2] for a guide to the literature). In this talk, we will consider algebras of partial functions for the signature consisting of the two following binary operations:

**Relative complement:** $f - g := \{(x, y) \mid (x, y) \in f \text{ and } (x, y) \notin g\}$,

**Domain restriction:** $f \rhd g := \{(x, y) \mid x \in \mathrm{dom}(f) \text{ and } (x, y) \in g\}$.

The choice of this signature was motivated by the following observations:

- we are able to express intersection: $f \cdot g := \{(x, y) \in (x, y) \in f \text{ and } (x, y) \in g\} = f - (f - g)$ (in particular, every algebra of partial functions is naturally equipped with a semilattice structure defined by $f \leq g \iff f \cdot g = f$),
- we are able to compare domains: $\mathrm{dom}(f) \subseteq \mathrm{dom}(g) \iff f \leq g \rhd f$.

[*] Speaker.

Formally, an **algebra of partial functions** of the signature $\{-, \triangleright\}$ is a $\{-, \triangleright\}$-subalgebra of $(\mathcal{PF}(X), -, \triangleright)$, where $\mathcal{PF}(X)$ denotes the set of all partial functions on a set $X$. **Representable algebras** are those $\{-, \triangleright\}$-algebras that are isomorphic to an algebra of partial functions. We will see that the class of representable algebras forms a finitely axiomatizable variety, and exhibit a representation for each such algebra. As a corollary we have that every representable algebra is representable by *injective* partial functions. Inside the class of representable algebras we will then investigate those that admit a **complete representation**, that is, an embedding into an algebra of partial functions turning existing *joins* into *unions* or, equivalently, turning existing *nonempty meets* into *intersections*. In particular, we will see that the completely representable algebras are precisely those algebras that are representable and *atomic*, and that this (universal-existential-universal) axiomatization is the simplest possible, in the sense that no existential-universal-existential axiomatization exists.

This is based on joint work with Brett McLean [1]. In the sequel to this paper, *Difference–restriction algebras of partial functions with operators: discrete duality and completion* [2], we present an adjunction (restricting to a duality) for the category of completely representable algebras and complete homomorphisms, which generalizes the adjunction between atomic Boolean algebras and sets. This is then extended to an adjunction/duality for completely representable algebras equipped with compatibility preserving completely additive operators.

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# On the concept of *Algebraic Crystallography*

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It is well known that, in the context of unital and strongly unital categories [1] and in the subtractive categories as well [7], [3], on any object $X$ there is at most one structure of abelian object. But, in these contexts, this did not seem so surprising because the three cases were closely related (strongly unital=unital+subtractive [5]) and because of the kind of their varietal origins: this uniqueness property arised naturally because, and when, some term in the definition of the varietal examples in question became a homomorphism in this variety.

Similar situations for other algebraic structures were even well known from a long time; for instance, it was clear that in a pointed Jónnson-Tarski variety, on any algebra $X$ there is at most one internal commutative monoid structure; the same property holds for the commutative and associative (=autonomous) Mal'tsev operations in the Mal'tsev varieties [6]. And again the limpid varietal contexts supplied the same simple explanation for this phenomenon.

But recently we were led to observe that the uniqueness structure for abelian objects still holds in the new context of Congruence hyperextensible categories [2]. This, in restrospect, emphasized that the uniqueness of the autonomous Mal'tsev operations was actually already noticed in Congruence Modular Varieties [4].

This phenomenon of uniqueness of some kinds of algebraic structures being now clearly extended to a much larger context than the one in the first paragraph, and the explanation by the existence of some kinds of terms in the definition of the varieties being no longer valid, it cannot remain possible to accept this uniqueness so easily and to keep it as an unquestioned process.

So, we propose to call *crystallographic for a given algebraic structure* any varietal or categorical setting in which, on any object $X$ of this setting, *there is at most one internal algebraic structure of this kind*, this terminology being chosen because, in such a setting, the algebraic structure in question is growing so scarce.

The aim of this talk will be to detail this situation in the context of Congruence hyperextensible varieties and categories, to establish the very first properties and general questionings about this *Algebraic Crystallography*, and finally to produce, from that, a spectacular observation with an example of an abelian category $Ab\mathbb{H}$ which i) fully faithfully embeds in a natural way the category $Ab$ of abelian groups and ii) in an independent way contains any category $K$-$Vect$ of $K$-vector spaces provided that the field $K$ is not of characteristic 2.

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# Admissibility of $\Pi_2$-Inference Rules: interpolation, model completion, and contact algebras

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The use of non-standard inference rules has a long tradition in modal logic starting from the pioneering work of Gabbay [4]. We consider a class of non-standard rules called $\Pi_2$-rules. An inference rule $\rho$ is a $\Pi_2$*-rule* if it is of the form

$$\frac{F(\underline{\varphi}/\underline{x}, \underline{y}) \to \chi}{G(\underline{\varphi}/\underline{x}) \to \chi}$$

where $F(\underline{x}, \underline{y}), G(\underline{x})$ are propositional formulas. We say that $\theta$ is obtained from $\psi$ by an application of the rule $\rho$ if $\psi = F(\underline{\varphi}/\underline{x}, \underline{y}) \to \chi$ and $\theta = G(\underline{\varphi}/\underline{x}) \to \chi$, where $\underline{\varphi}$ is a tuple of formulas, $\chi$ is a formula, and $\underline{y}$ is a tuple of propositional letters not occurring in $\underline{\varphi}, \chi$.

Rather little is known about the problem of recognizing *admissibility* for $\Pi_2$-rules. We show that there are tools already available in the literature on modal logic that can be fruitfully employed for studying admissibility of $\Pi_2$-rules. We present three different strategies for recognizing admissibility over a propositional modal system $\mathcal{S}$. In the following we will assume that $\rho$ is given by the formulas $F$ and $G$ as above.

## Conservative extensions, uniform interpolation, and model completions

Our first strategy applies to modal systems with the interpolation property. We determine admissibility of $\Pi_2$-rules via *conservative extensions*. We say that $\varphi(\underline{x}) \wedge \psi(\underline{x}, \underline{y})$ is a conservative extension of $\varphi(\underline{x})$ in $\mathcal{S}$ if for every formula $\chi(\underline{x})$, we have that $\vdash_{\mathcal{S}} \varphi(\underline{x}) \wedge \psi(\underline{x}, \underline{y}) \to \chi(\underline{x})$ implies $\vdash_{\mathcal{S}} \varphi(\underline{x}) \to \chi(\underline{x})$.

**Theorem 1.** *Assume that $\mathcal{S}$ has the interpolation property. A $\Pi_2$-rule $\rho$ is admissible in $\mathcal{S}$ iff $G(\underline{x}) \wedge F(\underline{x}, \underline{y})$ is a conservative extension of $G(\underline{x})$ in $\mathcal{S}$. In addition, if conservativity is decidable in $\mathcal{S}$, then $\Pi_2$-rules are effectively recognizable in $\mathcal{S}$.*

Our second strategy allows to determine the admissibility of $\Pi_2$-rules in systems with a universal modality. We use *uniform interpolation* which is a strengthening of ordinary interpolation. If $\varphi(\underline{x}, \underline{y})$ is a formula, its right global uniform pre-interpolant $\forall_{\underline{x}}\varphi(\underline{y})$ is a formula such that for every $\psi(\underline{y}, \underline{z})$ we have that

$$\psi(\underline{y}, \underline{z}) \vdash_{\mathcal{S}} \varphi(\underline{x}, \underline{y}) \quad \text{iff} \quad \psi(\underline{y}, \underline{z}) \vdash_{\mathcal{S}} \forall_{\underline{x}}\varphi(\underline{y}).$$

*Speaker.

**Theorem 2.** *Suppose that $\mathcal{S}$ has uniform global pre-interpolants and a universal modality $[\forall]$. Then a $\Pi_2$-rule $\rho$ is admissible in $\mathcal{S}$ iff*

$$\vdash_{\mathcal{S}} [\forall]\forall_{\underline{y}}(F(\underline{x},\underline{y}) \to z) \to (G(\underline{x}) \to z).$$

*Moreover, if $\mathcal{S}$ is decidable and global uniform interpolants are computable in $\mathcal{S}$, then $\Pi_2$-rules are effectively recognizable in $\mathcal{S}$.*

Our third strategy exploits the connection between $\Pi_2$-rules and model-theoretic machinery. With each $\Pi_2$-rule $\rho$, we associate the following $\forall\exists$-statement in the *first-order* language of $\mathcal{S}$-algebras:

$$\Pi(\rho) := \forall \underline{x}, z\Big(G(\underline{x}) \not\leq z \Rightarrow \exists \underline{y} : F(\underline{x},\underline{y}) \not\leq z\Big).$$

**Theorem 3.** *Suppose that $\mathcal{S}$ has a universal modality and let $T_{\mathcal{S}}$ be the theory of the simple non-degenerate $\mathcal{S}$-algebras. If $T_{\mathcal{S}}$ has a model completion $T^{\star}_{\mathcal{S}}$, then a $\Pi_2$-rule $\rho$ is admissible in $\mathcal{S}$ iff $T^{\star}_{\mathcal{S}} \models \Pi(\rho)$.*

As a consequence, we obtain an alternative way to recognize admissibility.

**Corollary 4.** *Let $\mathcal{S}$ be a system with universal modality that is decidable and locally tabular. If simple $\mathcal{S}$-algebras enjoy the amalgamation property, then admissibility of $\Pi_2$-rules in $\mathcal{S}$ is effective.*

## Contact algebras and admissibility in $\mathsf{S}^2\mathsf{IC}$.

Recently, there has been a renewed interest in non-standard rules in the context of the region-based theories of space. One of the key algebraic structures in these theories is that of *contact algebras*. Compingent algebras are contact algebras satisfying two $\forall\exists$-sentences (aka $\Pi_2$-sentences). De Vries [3] established a duality between complete compingent algebras and compact Hausdorff spaces. This duality led to new logical calculi for compact Hausdorff spaces in [1, 2]. Key to these approaches is a development of logical calculi corresponding to contact algebras. In [2] such a calculus is called the *strict symmetric implication calculus* and is denoted by $\mathsf{S}^2\mathsf{IC}$. The extra $\Pi_2$-axioms of compingent algebras then correspond to non-standard $\Pi_2$-rules, which turn out to be admissible in $\mathsf{S}^2\mathsf{IC}$.

We apply our third strategy to study admissibility of $\Pi_2$-rules in $\mathsf{S}^2\mathsf{IC}$. We also show that the admissibility problem for $\mathsf{S}^2\mathsf{IC}$ is co-NExpTime-complete. This is done by using the model completion of the theory of contact algebras. Moreover, we explicitly list three sentences that, together with the axioms of contact algebras, axiomatize the model completion.

**Theorem 5.** *The model completion of the theory of contact algebras is finitely axiomatizable*

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# Hereditary Structural Completeness over K4

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In deductive systems a rule is said to be *admissible* if the tautologies of the system are closed under its applications and *derivable* if the rule itself holds in the system [5]. Whilst every derivable rule for a system is admissible whether the converse holds, varies between deductive systems. When it does we say the system is *structurally complete*, as one might expect the classical propositional calculus (CPC) is structurally complete, but many non-classical systems including the intuitionistic propositional calculus (IPC) are not [1]. A classical problem is to determine which deductive systems are *structurally complete*. Early investigations suggested it would be possible to precisely characterise the *hereditarily structurally complete* (HSC) systems, those which are not only themselves structurally complete but whose finitary extensions are too. This proved a fruitful question, Citkin [3] produced a characterisation for intermediate logics and Rybakov [6, 7] did so for transitive modal logics. Both these characterisations take a similar form.

**Citkin's Theorem** An intermediate logic is HSC iff the variety of Heyting algebras associated with it omits five finite algebras [3].

**Rybakov's Theorem** A transitive modal logic is HSC iff it is not included in the logic of a list of 20 frames [7, pg 274].

Recently, Bezhanishvili and Moraschini [1] gave a new proof of Citkin's theorem. Their approach draws upon both abstract algebraic logic and duality theory. Techniques from abstract algebraic logic allow one to establish that an algebrizable logic is HSC iff its associated variety of algebras is primitive [1, Section 2], that is every all its sub quasi-varieties are in fact varieties. IPC is algebrizable by the variety of Heyting algebra and consequently the task of characterising hereditary structurally complete intermediate logics is equivalent to that of characterising primitive subvarieties of Heyting algebras[1, Section 2]. Results from universal algebra further reduce the problem to centre around the notion of weak projectivity. An algebra $A$ is *weakly projective* in a variety $V$ iff for every $B \in V$ if $A$ is a homomorphic image of $B$ then $A$ is isomorphic to a subalgebra of $B$.

**Lemma 1** Let $V$ be a locally finite variety, that is all its finitely generated members are finite. Then $V$ is primitive iff its finite, non-trivial, finitely subdirectly irreducible (FSI) members are weakly projective in $V$.

The investigation is further aided through the Esakia duality between Heyting algebras and Esakia spaces[1, Section 3]. This allows the reduced algebraic question to be investigated with topological methods.

Notably a similar framework exists for transitive modal logics; they are algebrizable by the variety of K4-algebras [4] which are linked by Jónnson-Tarski duality to the class of transitive modal spaces. This allows us to do for Rybakov's result what Bezhanishvili and Moraschini did for Citkin's and investigate HSC modal logics through K4-algebras and transitive modal spaces.

[*] Speaker.

More than simply provide a new proof of Rybakov's theorem, this approach illuminates a mistake in Rybakov's characterisation. The list of frames given by Rybakov is too restrictive and the characterisation of HSC transitive modal logics is revised accordingly.

**Theorem 2** The variety generated by the algebra dual to $F_3'$ is primitive, where $F_3'$ is the transitive space $(\{x, y, z\}, \tau, R)$ where $R = \{(x, y), (x, z), (y, z), (z, z)\}$ and $\tau$ is the discrete topology.
**Revised Theorem** A transitive modal logic is HSC iff the variety of K4-algebras associated with it omits the algebras $(F_i)^* : 1 \leq i \leq 17$ and omit the algebra $(G_n)^*$ for some $n \in \omega$.

The proof strategy for the new revised system is the same. However, varieties of K4-algebras are not necessarily locally finite so an alternative to lemma 1 is needed.

**Lemma 3** Let $V$ be a variety of K4-algebras. If $V$ is primitive then the finite, non-trivial FSI members of $V$ are weakly projective in $V$. Moreover, suppose all sub-varieties of $V$ have the finite model property (FMP). Then if the finite, non-trivial FSI members of $V$ are weakly projective in $V$ then $V$ is primitive.

Consequently, the poof strategy for the revised theorem has four components. The first is to establish the easier direction of the revised theorem.

**Lemma 4** Primitive varieties of K4-algebras omit the algebras $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n \in \omega$.

The second harder direction is much more involved. A crucial step is to give a precise description of the finitely generated, non-trivial, subdirectly irreducible (SI) members of varieties of K4-algebras omitting the given algebras. This description then drives the proofs of the final two key results.

**Lemma 5** Let $V$ be a variety of K4-algebras omitting $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n \in \omega$. Then $V$ has the FMP.

**Lemma 6** Let $V$ be a variety of K4-algebras omitting $(F_i)^* : 1 \leq i \leq 17$ and $(G_n)^*$ for some $n \in \omega$. Every finite, non-trivial FSI member of $V$ is weakly projective in $V$.

Combing lemmas 3, 4 and 5 then yields a proof of the new revised theorem.

This work is a summary of a master's thesis undertaken at the Institute for Logic, Language and Computation [2].

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# (A bit more) abstract Lindenbaum lemma

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The relation between consequence relations, closure operators and closures systems is well known and so is the notion of *basis* of a closure systems (a family of closed sets which allows one to express *any* closed set as an intersection of some of its subfamily). The logical relevance of this notion is embodied in the *Lindenbaum lemma* which says that maximally consistent theories form a basis of the system of all theories (i.e., deductively closed sets) of classical propositional logic (which can be equivalently formulated as saying that every consistent theory can be extended into a maximal consistent one).

In the setting of non-classical logics the maximally consistent theories are not always sufficient to obtain the result; one has to look at, for example, the prime/complete/linear theories (depending on the logic in question). While these classes of theories are usually defined using certain logical connectives (in the mentioned examples by disjunction/negation/implication) they can usually be defined abstractly as (finitely) meet-irreducible ones.As the structurality of the underlying consequence relation is irrelevant for such a notion (i.e., it can be defined for a closure system over an arbitrary set of elements), one can formulate the following well-known crucial result of (not only) Algebraic logic:

**Abstract Lindenbaum lemma** Let $\mathcal{C}$ be a closure system associated to a finitary consequence relation. Then the meet-irreducible closed sets form a basis of $\mathcal{C}$.

While the finitarity restriction is crucial for its usual proof, it is not necessary: there are works (e.g. [3, 4, 5, 6, 7, 8]) proving it (or its variant for *finitely* meet-irreducible theories) for certain *infinitary structural* consequence relations (usually modal, dynamic, or fuzzy logics). The paper [1] provides a general result (covering most of the known cases) for *structural* consequence relations with a *countable* Hilbert-style axiomatization and a *strong disjunction* (see [2] for more details).

The main contribution of this paper is identifying of non-structural formulations of the necessary properties of that result and subsequent proof of its truly abstract version: We say that a consequence relation $\vdash$ on a set $A$ with associated closure operator $C$ and closure system $\mathcal{C}$

- is *framal,* if $\mathcal{C}$ is a frame, i.e., for each $\{X\} \cup \mathcal{Y} \subseteq \mathcal{C}$,
$$X \cap \bigvee \mathcal{Y} = \bigvee_{Y \in \mathcal{Y}} (X \wedge Y).$$
- has the *finitely generated intersection property* if for any finite sets $X, Y$ there is a finite set $U$ such that:
$$C(X) \cap C(Y) = C(U).$$
- is *countably axiomatizable* if there is a countable system $\mathcal{AS} \subseteq \mathcal{P}(A) \times A$ such that $X \vdash x$ iff there is tree without infinite branches labeled by elements of $A$ such that
  - its root is labeled by $x$,
  - if $y$ is a label of some of its leafs, then $y \in X$ or $\langle \emptyset, y \rangle \in \mathcal{AS}$,
  - if a non-leaf is labeled by $y$ and $Y$ is the set of labels of its direct predecessors, then $\langle Y, y \rangle \in \mathcal{AS}$.

**Abstract Lindenbaum lemma for infinitary logics** Let $\mathcal{C}$ be a closure system on a countable set $A$ associated to a countably axiomatizable framal consequence relation with finitely generated intersection property. Then the finitely meet-irreducible closed sets form a basis of $\mathcal{C}$.

None of the three assumptions on the consequence relation can be omitted, indeed we can present examples satisfying any pair of these conditions and failing the Lindenbaum lemma (and thus also the final condition).

Let us end with a sketch of the proof. Its main tool is a binary relation $\Vdash$ on $\mathcal{P}(A)$ defined for an arbitrary consequence relation on $A$ with associated closure operator $C$ as:

$$X \Vdash Y \quad \text{iff} \quad \text{there is finite } Y' \subseteq Y \text{ such that } \bigcap_{y \in Y'} C(y) \subseteq C(X).$$

The two crucial facts about $\Vdash$ are ($\mathcal{C}$ is the associated closure system to $C$):

- If $\mathcal{C}$ is a frame, then for each sets $X, P \subseteq A$ and each *finite* set $Y \subseteq A$ we have:

$$\frac{\{X \Vdash Y \cup \{p\} \mid p \in P\} \qquad\qquad X \cup P \Vdash Y}{X \Vdash Y}.$$

- If $X \nVdash Y$ and $X \cup Y = A$, then $X$ is a finitely meet-irreducible element of $\mathcal{C}$.

The proof is done by finding, for a given $x \notin C(X)$, a finitely meet-irreducible $X' \in \mathcal{C}$ such that $X \subseteq X'$ and $x \notin X'$. We start by enumerating all elements of an *existing countable axiomatic system* $\mathcal{AS}$ and construct a sequence of pairs $\langle X_i, Y_i \rangle$ where $Y_i$ is finite and $X_i \nVdash Y_i$. Starting with $\langle X, \{x\} \rangle$ we in each step use the cut-like rule mentioned above (recall that we assume that $\mathcal{C}$ *is a frame*) to process the rule $\langle P_i, c_i \rangle \in \mathcal{AS}$ making sure that (roughly speaking) either we "do not have to use it" by adding $c_i$ to $X_i$ or that we "cannot use it" by adding some element of $P_i$ to $Y_i$. Taking $X'$ and $Y'$ as unions of the corresponding sequences we show that $y \in C(X')$ iff $y \in X_i$ for some $i$ which entails, using the *finitely generated intersection property,* that $X' \nVdash Y'$. Assuming that our axiomatic systems contains dummy rules $\langle \{z\}, z \rangle$ for each $z$ we also obtain $X' \cup Y' = A$ and thus we know that $X'$ is the set we are looking for.

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# Translational Embeddings via Stable Canonical Rules

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This paper, based on [3], presents a new uniform method for studying modal companions of superintuitionistic (si) deductive systems and related notions, based on the machinery of stable canonical rules developed, e.g., in [1]. Our techniques recover much of the existing theory of modal companions, expand it with new results, and generalize smoothly to rule systems admitting filtrations in richer signatures.

A *si-rule* (*modal rule*) is a pair $\Gamma/\Delta$ with $\Gamma, \Delta$ finite sets of si (modal) formulae. *Si-* and *normal modal rule systems* (defined in [1]) are sets of si- or modal rules axiomatizing universal classes of Heyting algebras and modal algebras respectively, the way si-logics and normal modal logics axiomatize varieties of Heyting and modal algebras. Let $\mathbf{Ext}(\mathtt{IPC})$ and $\mathbf{NExt}(\mathtt{K})$ be the lattices of si- and normal modal logics respectively. For each $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$ there is a least si-rule system $\mathtt{L_R}$ containing $\varnothing/\varphi$ for each $\varphi \in \mathtt{L}$, and similarly for normal modal logics. Thus the maps $\mathtt{L} \mapsto \mathtt{L_R}$ and $\mathtt{M} \mapsto \mathtt{M_R}$ are embeddings of $\mathbf{Ext}(\mathtt{IPC})$ and $\mathbf{NExt}(\mathtt{K})$ into the lattices of si-rule systems $\mathbf{Ext}(\mathtt{IPC_R})$ and of normal modal rule systems $\mathbf{NExt}(\mathtt{K_R})$ respectively.

The *Gödel translation* $T(\varphi)$ of a si-formula $\varphi$ is obtained by prefixing every subformula of $\varphi$ with $\Box$. Lift the Gödel translation to rules by setting $T(\Gamma/\Delta) := T[\Gamma]/T[\Delta]$. For $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC})$, set $\tau(\mathtt{L}) := \mathtt{S4} \oplus \{T(\varphi) : \varphi \in \mathtt{L}\}$ and $\sigma(\mathtt{L}) := \mathtt{Grz} \oplus \tau(\mathtt{L})$, and similarly for si-rule systems. For $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4})$, set $\rho(\mathtt{M}) := \{\varphi : T(\varphi) \in \mathtt{M}\}$, and similarly for normal modal rule systems. A normal modal logic (rule system) $\mathtt{M}$ is a *modal companion* of a si-logic (rule system) $\mathtt{L}$ if $\rho(\mathtt{M}) = \mathtt{L}$.

A map $f : \mathfrak{X} \to \mathfrak{Y}$ between Esakia spaces $\mathfrak{X}, \mathfrak{Y}$ is *stable* if continuous and relation preserving. If $\mathfrak{D} \subseteq \wp(Y)$, a map $f : \mathfrak{X} \to \mathfrak{Y}$ satisfies the *bounded domain condition* (BDC) for $\mathfrak{D}$ when for any $x \in X$ and $\mathfrak{d} \in \mathfrak{D}$, if ${\uparrow} f(x) \cap \mathfrak{d} \neq \varnothing$ then $f[{\uparrow} x] \cap \mathfrak{d} \neq \varnothing$, where ${\uparrow} x := \{y : x \leq y\}$. Analogously, stable maps and the BDC are defined for modal spaces. For every finite Esakia space $\mathfrak{F}$ and any $\mathfrak{D} \subseteq \wp(F)$ there is a *si-stable canonical rule* $\eta(\mathfrak{F}, \mathfrak{D})$ which is refuted in an Esakia space $\mathfrak{X}$ iff there is a stable surjection $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$. Similarly, every finite modal space $\mathfrak{F}$ and any $\mathfrak{D} \subseteq \wp(F)$ induce a *modal stable canonical rule* $\mu(\mathfrak{F}, \mathfrak{D})$ which is refuted in a modal space $\mathfrak{X}$ iff there is a stable surjection $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$ [1]. All si- and normal modal rule systems are axiomatizable by stable canonical rules.

Our first main result is an alternative proof of the following theorem.

**Theorem 1.** *The following pairs of maps are mutually inverse complete lattice isomorphisms:*

1. $\sigma : \mathbf{Ext}(\mathtt{IPC_R}) \to \mathbf{NExt}(\mathtt{Grz_R})$ *and* $\rho : \mathbf{NExt}(\mathtt{Grz_R}) \to \mathbf{Ext}(\mathtt{IPC_R})$ *[2].*
2. $\sigma : \mathbf{Ext}(\mathtt{IPC}) \to \mathbf{NExt}(\mathtt{Grz})$ *and* $\rho : \mathbf{NExt}(\mathtt{Grz}) \to \mathbf{Ext}(\mathtt{IPC})$ *[4].*

If $\mathfrak{X}$ is a closure space, its *skeleton* $\rho\mathfrak{X}$ is the Esakia obtained by collapsing clusters in $\mathfrak{X}$ and setting $\{\rho[U] : U \in \mathsf{Clop}(\mathfrak{X})\}$ as a basis, where $\rho : \mathfrak{X} \to \rho\mathfrak{X}$ is the map sending each $x \in \mathfrak{X}$ to its cluster. We let $\sigma\rho\mathfrak{X}$ be $\rho\mathfrak{X}$, viewed as a closure space. Theorem 1 follows from lemma 2 below, which we establish using the refutation conditions of stable canonical rules.

**Lemma 2.** *Let $\mathfrak{X}$ be a $\mathtt{Grz}$-space. Then for every modal rule $\Gamma/\Delta$, $\mathfrak{X} \models \Gamma/\Delta$ iff $\sigma\rho\mathfrak{X} \models \Gamma/\Delta$.*

*Proof sketch.* ($\Rightarrow$) is easy. To prove ($\Leftarrow$), we assume wlog that $\Gamma/\Delta = \mu(\mathfrak{F}, \mathfrak{D})$, for $\mathfrak{F}$ a finite closure space. If $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathcal{D})$, then there is a stable surjection $f : \mathfrak{X} \to \mathfrak{F}$ satisfying the BDC for $\mathfrak{D}$. Let $C = \{x_1, \ldots, x_n\} \subseteq F$ be some cluster. By the properties of $\mathtt{Grz}$-spaces, there are disjoint $U_1, \ldots, U_n \in \mathsf{Clop}(\sigma\rho\mathfrak{X})$ with $\rho[M_i] \subseteq U_i$ and $\bigcup_i U_i = \rho[Z_C]$, where $M_i := max(f^{-1}(x_i))$. Thus for each cluster $C \subseteq F$ we may define a map $g_C : \rho[Z_C] \to C$ by setting $z \mapsto x_i \iff z \in U_i$. We combine these into a map $g : \sigma\rho\mathfrak{X} \to F$ by setting $g(\rho(z)) := g_C(\rho(z))$ if $f(z) \in C$ for some proper cluster $C$, and $g(\rho(z)) := f(z)$ otherwise. It can be shown that $g$ is a stable surjection satisfying the BDC for $\mathfrak{D}$, which establishes $\sigma\rho\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$. ☐

We also axiomatically characterize the modal companion maps via stable canonical rules.

**Theorem 3.** *Let* $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ *be such that* $\mathtt{L} = \mathtt{IPC_R} \oplus \{\eta(\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}$. *Then we have:*

1. $\tau\mathtt{L} = \mathtt{S4_R} \oplus \{\mu(\sigma\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}$
2. $\sigma\mathtt{L} = \mathtt{Grz_R} \oplus \{\mu(\sigma\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}$.

**Theorem 4.** *Let* $\mathtt{M} \in \mathbf{NExt}(\mathtt{S4_R})$ *with* $\mathtt{M} = \mathtt{S4_R} \oplus \{\mu(\mathfrak{F}_i, \mathfrak{D}_i) : i \in I\}$, *and let* $\rho\mathfrak{D} := \{\rho[\mathfrak{d}] : \mathfrak{d} \in \mathfrak{D}\}$. *Then we have:*

$$\rho\mathtt{M} = \mathtt{IPC_R} \oplus \{\eta(\rho\mathfrak{F}_i, \rho\mathfrak{D}_i) : \mu(\sigma\rho\mathfrak{F}_i, \rho\mathfrak{D}_i) \in \mathtt{M}\}.$$

Theorem 3 follows from the fact that for all si-stable canonical rules $\eta(\mathfrak{F}, \mathfrak{D})$ we have that $T(\eta(\mathfrak{F}, \mathfrak{D}))$ is equivalent to $\mu(\sigma\mathfrak{F}, \mathfrak{D})$ (*rule translation lemma*). We prove Theorem 4 by showing that for any modal stable canonical rule $\mu(\mathfrak{F}, \mathfrak{D})$ with $\mathfrak{F}$ a preorder and for any closure space $\mathfrak{X}$, if $\mathfrak{X} \not\models \mu(\mathfrak{F}, \mathfrak{D})$ then $\rho\mathfrak{X} \not\models \eta(\rho\mathfrak{F}, \rho\mathfrak{D})$ (*rule collapse lemma*).

Lastly, we generalize the Dummett-Lemmon conjecture [5, Corollary 2] to rule systems.

**Theorem 5.** *A si-rule system* $\mathtt{L} \in \mathbf{Ext}(\mathtt{IPC_R})$ *is Kripke complete iff* $\tau\mathtt{L}$ *is.*

*Proof sketch.* ($\Leftarrow$) is easy. To prove ($\Rightarrow$), let $\mathtt{L}$ be Kripke complete. Suppose that $\Gamma/\Delta \notin \tau\mathtt{L}$. Wlog, we assume $\Gamma/\Delta = \mu(\mathfrak{F}, \mathfrak{D})$ for $\mathfrak{F}$ a preorder. By rule collapse lemma, $\eta(\rho\mathfrak{F}, \rho\mathfrak{D}) \notin \mathtt{L}$. Since $\mathtt{L}$ is Kripke complete, there is a si Kripke frame $\mathfrak{Y}$ and a stable surjection $f : \mathfrak{Y} \to \rho\mathfrak{F}$ satisfying the BDC for $\rho\mathfrak{D}$. For every $x \in \rho[F]$ look at $\rho^{-1}(x)$, let $k = |\rho^{-1}(x)|$ and enumerate $\rho^{-1}(x) = \{x_1, \ldots, x_k\}$. Working in $\mathfrak{Y}$, for every $y \in f^{-1}(x)$ replace $y$ with a $k$-cluster $y_1, \ldots, y_k$ and extend the relation $R$ clusterwise. The result, $\mathfrak{Z}$, is a Kripke frame with $\mathfrak{Z} \models \tau\mathtt{L}$. We identify $\rho\mathfrak{Z} = \mathfrak{Y}$. For every $x \in \rho[F]$ define a map $g_x : f^{-1}(x) \to \rho^{-1}(x)$ by setting $g_x(y_i) = x_i$ $(i \leq k)$. Finally, define $g : \mathfrak{Z} \to \mathfrak{F}$ by putting $g = \bigcup_{x \in \rho[F]} g_x$. It can be shown that $g$ is a stable surjection satisfying the BDC for $\mathfrak{D}$, thus establishing $\mathfrak{Z} \nvDash \mu(\mathfrak{F}, \mathfrak{D})$. ☐

Via uniform generalizations of our techniques, we obtain similar results in the settings of bi-superintuitionistic and tense deductive systems, and of deductive systems over the modal intuitionistic logic of provability $\mathtt{KM}$ and classical provability logic $\mathtt{GL}$. For details, consult [3].

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# Deduction via 2-category theory

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The purpose of this talk is to present a category-based unified approach that accommodates diverse takes on the topic of *deduction*. The effort required in order to do so turns out to be extremely fruitful, and in fact it can be used, for example, to obtain novel results about the algebraic treatment of type constructors in dependent type theory.

One of the motivating examples is to give a theoretical framework in which the two following rules, which stand on very conceptually different grounds, can be compared.

$$\text{(Subs)}\ \frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B}{\Gamma \vdash B[a]} \qquad\qquad \text{(Cut)}\ \frac{x;\Gamma \vdash \phi \qquad x;\Gamma,\phi \vdash \psi}{x;\Gamma \vdash \psi}$$

One can traditionally be found in type theory [6], the other in proof theory [8]: despite their incredibly similar look, and the somehow parallel development of the respective theories in the same notational framework, there are some philosophical differences between the interpretation of the symbols above. Not only that, but the same "$\vdash$" symbol seems to regard only statements of one kind `formula` in the case of (Cut), while it pertains to two - `term` and `type` - in that of (Subs).

Of course one could argue that these different points of view are mostly philosophical, and, in particular, the deep connection between proof theory and type theory has been studied for a while: its development falls under the paradigm that is now mostly known as *propositions-as-types* [9]. We believe our theory gives testament to that and, in fact, it gives it a categorical backbone.

Rebooting some ideas from [5], we develop what we call *judgemental theories*. Going back to the example of (Subs) and (Cut), we intuitively see how they both fit the same paradigm, in the sense that we could read both as instances of the following syntactic string of symbols

$$(\triangle)\ \frac{\heartsuit \vdash \blacksquare \qquad \square \vdash \clubsuit}{\heartsuit \vdash \spadesuit}$$

which we usually parse as: *by* $\triangle$, *given* $\heartsuit \vdash \blacksquare$ *and* $\square \vdash \clubsuit$ *we deduce* $\heartsuit \vdash \spadesuit$. Our theory allows for a coherent expression of all such strings of symbols, and shows how a suitable choice of *context* either produces (Subs) or (Cut): it is not about the interpretation of the symbols, just about the relation they are in with one another.

Concretely, a judgemental theory is a 2-subcategory of **Cat** closed under some constructions which aim to encode deductive power into the system, such as finite limits and lifting of 2-cells along fibrations, but everything that we develop can be inherently repeated into any 2-category. Each "kind" of entailment/context relation ($\vdash$) is represented by a functor - often, a fibration - over a fixed context category. Each rule is represented by a (lax) commutative triangle - often, a morphism of fibrations - involving such functors. Starting from a bunch of such choices, we show that a few categorical constructions allow us to produce new (lax) commutative triangles,

*Speaker.

hence new rules. In fact, they produce all structural rules both in the case of dependent type theory and of natural deduction.

Being very syntactic in nature, our framework has both the advantage of being versatile and computationally meaningful. It allows, for example, to give a general definition of type constructor, a feature that has not been available before.

If the process of formalization of a given deductive system is purely syntactical, in the sense that we are not interested in what a given judgement or rule should *mean*, only in the symbols involved, the judgemental theory we obtain is often as well behaved as one hopes a categorical semantics would be: we consider the case study of dependent types, and show how traditional categorical models ([5], [7], [4], [3], [2]) all fit into our paradigm. Moreover, properties that were considered *external*, such has having dependent sums for CE-systems [1], are *internalized* in our framework, so that one can quantitatively compare different models.

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# Equidivisibility and profinite coproduct

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A semigroup is *equidivisible* if any two factorizations of an arbitrary element of the semigroup have a common refinement. The property of a semigroup being equidivisible was introduced and studied in [6] as a natural common generalization of free semigroups and groups.

More recently, this property appeared frequently as a useful tool in the study of relatively free profinite semigroups. A profinite semigroup is relatively free if it is a free object in the category of pro-$\mathsf{V}$ semigroups, for some pseudovariety $\mathsf{V}$ of finite semigroups. (A pseudovariety of semigroups is a class of finite semigroups closed under taking homomorphic images, subsemigroups, and finite products.)

In [1, 5] it was shown that if the pseudovariety $\mathsf{V}$ is such that the product of any two $\mathsf{V}$-recognizable languages is still $\mathsf{V}$-recognizable, then the finitely generated free pro-$\mathsf{V}$ semigroups are equidivisible. This includes the cases where $\mathsf{V}$ is the pseudovariety of all finite semigroups, and where $\mathsf{V}$ is the pseudovariety of all finite aperiodic semigroups. Other recent papers where the equidivisibility of relatively free profinite semigroups is applied or deserves attention include [4, 3, 8].

A complete characterization of the pseudovarieties for which the corresponding finitely generated relatively free profinite semigroups are equidivisible (dubbed *equidivisible pseudovarieties*) appears in [2]. This characterization is done via a functor on the category of finitely generated semigroups, called the *two-sided Karnofsky–Rhodes expansion*. Given an onto homomorphism $\varphi\colon A^+ \to S$, its two sided Karnofsky–Rhodes expansion is the onto homomorphism $\varphi^{\mathsf{KR}}\colon A^+ \to S_\varphi^{\mathsf{KR}}$ that identifies words with the same image under $\varphi$ and such that the naturally associated paths of the two-sided Cayley graph of $\varphi$ have the same transition edges. The semigroup $S_\varphi^{\mathsf{KR}}$ is said to be a two-sided Karnofsky–Rhodes expansion of $S$.

**Theorem 1** ([2])**.** *A finitely generated profinite semigroup is equidivisible if and only if it is contained in the pseudovariety of completely simple semigroups or it is closed under two-sided Karnofsky–Rhodes expansion.*

As observed in [6], the class of equidivisible semigroups is closed under taking free products, that is, coproducts in the category of semigroups. Here we present an analog for profinite semigroups. For that purpose, we introduce $\mathsf{V}$-coproducts of pro-$\mathsf{V}$ semigroups with respect to a pseudovariety of semigroups $\mathsf{V}$, extending what was done in [7] for the pseudovariety of finite groups. We give simple conditions on $\mathsf{V}$ guaranteeing that the free product of pro-$\mathsf{V}$ semigroups embeds naturally in their $\mathsf{V}$-coproduct.

The following definition is the key to obtain our main new results.

*Speaker.

**Definition 2** (KR-cover of a finite semigroup)**.** Let $S$ be a profinite semigroup, and let $T$ be a finite semigroup. We say that $S$ is a *KR-cover of* $T$ when $T$ is a continuous homomorphic image of $S$ and for every continuous onto homomorphism $\varphi : S \to T$ there is a generating mapping $\psi : A \to T$, for some finite alphabet $A$ depending on $\varphi$, and a continuous homomorphism $\varphi_\psi : S \to T_\psi^{\mathsf{KR}}$ such that the following diagram commutes:

$$\begin{tikzcd} & S \arrow[dl, dashed, "\varphi_\psi"'] \arrow[d, "\varphi"] \\ T_\psi^{\mathsf{KR}} \arrow[r, "\pi_\psi"'] & T. \end{tikzcd}$$

A profinite semigroup $S$ is a *KR-cover* if it is a KR-cover of each of its finite continuous homomorphic images.

As examples of KR-covers, we have the profinite groups and the free profinite semigroups relatively to a pseudovariety closed under taking two-sided Karnofsky–Rhodes expansion. We show that every KR-cover is equidivisible. One of our mains results is the following:

**Theorem 3.** *For every pseudovariety of semigroups* $\mathsf{V}$ *closed under two-sided Karnofsky–Rhodes expansion, the class of all pro-*$\mathsf{V}$ *KR-covers is closed under* $\mathsf{V}$*-coproducts.*

This theorem allows us to build new examples of equidivisible profinite semigroups from old ones. However, there are examples of finite equidivisible semigroups that are not KR-covers.

Let $A$ be a finite alphabet, and $\mathsf{V}$ be a pseudovariety closed under two-sided Karnofsky–Rhodes expansion. Using equidivisibility, one sees that the $A$-generated relatively free profinite semigroup over $\mathsf{V}$, denoted $\overline{\Omega}_A\mathsf{V}$, has the following cancellation property: if $au = bv$ or $ua = vb$, with $a, b \in A$ and $u, v \in \overline{\Omega}_A\mathsf{V}$, then $a = b$ and $u = v$. Abstracting this property, we get the class of the so called *letter super-cancellative* profinite semigroups. It turns out that within this class the KR-covers completely characterize the equidivisible profinite semigroups. That is, we have:

**Theorem 4.** *Let* $S$ *be a finitely generated profinite semigroup that is letter super-cancellative. Then* $S$ *is equidivisible if and only if it is a KR-cover.*

We build examples of letter super-cancellative equidivisible profinite semigroups that are not relatively free profinite semigroups.

With the previous theorem on hand, we are able to deduce the following result.

**Theorem 5.** *For every pseudovariety of semigroups* $\mathsf{V}$ *closed under two-sided Karnofsky–Rhodes expansion, the class of letter super-cancellative equidivisible finitely generated pro-*$\mathsf{V}$ *semigroups is closed under finite* $\mathsf{V}$*-coproducts.*

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# Non-distributive logics as evidential logics

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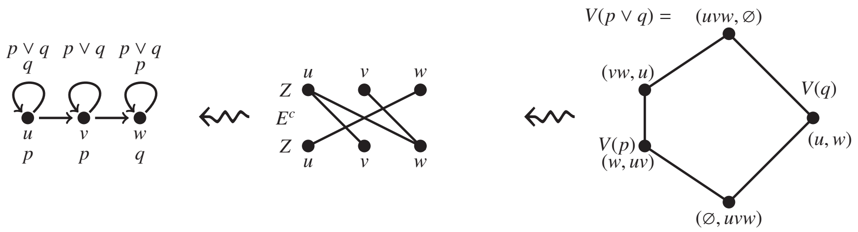
The term 'non-distributive logics'(cf. [1]) refers to the wide family of non-classical propositional logics in which the distributive laws $\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and $(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \vdash \alpha \vee (\beta \wedge \gamma)$ do not need to be valid. Since the rise of very well known instances such as quantum logic [9], interest in non-distributive logics has been building steadily over the years. Techniques and ideas have come from pure mathematical areas such as lattice theory, duality and representation (cf. [8, 6]), and areas in mathematical logic such as algebraic proof theory (cf. [5, 2]), but also from the philosophical and formal foundations of quantum physics [7, 1], philosophical logic [11] theoretical computer science and formal linguistics [10].

In this talk, we present a type of (Kripke-style) relational semantics for non-distributive logics which is based on reflexive directed graphs (i.e. tuples $(Z, E)$ such that $Z$ is a set and $E \subseteq Z \times Z$ is reflexive), as in the left-hand side of the picture below. Via an intermediate structure, every such graph can be associated with a complete lattice, as in the right-side of the picture. Thanks to this fact, the interpretation of non-distributive logics on lattice-based algebras transfers to graph-based relational models. The topic we will discuss in this presentation is part of an ongoing line of research [4], whose developments are technically rooted in dual characterization results and insights from unified correspondence theory.




Interestingly, the distinguishing feature of this *graph-based semantics* is that, at any given state $z$ of any such model, a formula $\phi$ can be satisfied ($z \Vdash \phi$), refuted ($z \succ \phi$), or *neither*. We will argue that, thanks to this feature, graph-based models support an interpretation of non-distributive logics as *evidential logics*, i.e. logics aimed at capturing correct reasoning in situations in which the notion of truth and falsity is based on the availability of evidence (in support or against a proposition). These notions of truth and falsity are even more refined than their intuitionistic analogues, since, in order to refute a formula, it is not enough there being lack of evidence supporting it, but rather, evidence against it needs to be presented.

In this talk we will show that a systematic relationship can be established between the first-order correspondents of all Sahlqvist modal reduction principles[1] on Kripke frames and graph-based frames. For instance, the Sahlqvist modal reduction principle $\Box p \vdash p$, which corresponds to the reflexivity condition $\Delta \subseteq R$ on Kripke frames $(W, R)$, corresponds to the first-order condition $E \subseteq R$ on graph-based frames $(Z, E, R)$.

More in general, the first order correspondents of Sahlqvist modal reduction principles for graph-based semantics can be formulated as the $E$-counterparts of their first-order correspondents on Kripke frames. This gives rise to the notion of *parametric correspondence* [3] in graph-based frames.

[1] *Sahlqvist modal reduction principles* are sequents of the form $\phi[\alpha(p)/x] \vdash \psi[\chi(p)/y]$ or $\phi[\chi(p)/x] \vdash \psi[\beta(p)/y]$, where $\phi(x)$ and $\beta(p)$ are built out of $\Diamond$ connectives, $\psi(x)$ and $\alpha(p)$ out of $\Box$ connectives, and $\chi(p)$ out of both $\Box$ and $\Diamond$ connectives.

Besides being of technical interest, this result lends itself as a base for further and more conceptual investigations on how a given interpretation of a modal axiom transfers from one semantic context to another. For instance, We show that the first order correspondents of Sahlqvist modal reduction principles on graph-based semantics can be seen as *lifted* versions of their first order correspondents on the Kripke frames under a suitable notion of lifting. On the other hand, first order correspondents on graph-based frames reduce to the first correspondents on Kripke-frames when the relation defining them, that is, the "parameter", is identity.

When comparing the meaning of $\Box p \vdash p$ on Kripke models and on graph-based models under the epistemic understanding of $\Box$, the factivity reading of the axiom corresponds to the reflexivity condition on Kripke models requiring the agent to not exclude the true world. Similarly, the $E$-counterpart of reflexivity on a graph-based frame requires the agent to not exclude any world which is an $E$-successor of the true one, which corresponds to factivity in a setting in which different states of affairs might be *inherently indiscernible* (and their inherent indiscernibility is encoded by the relation $E$).

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# Algorithmic correspondence and analytic rules for (D)LE logics

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A core line of research in structural proof theory focuses on the algorithmic or semi-algorithmic generation of analytic rules (we refer to [1] for a detailed survey of the relevant literature).

In [15, 12, 13, 14], a class of first order formulas, referred to as *(co-)geometric formulas*, is identified and used for effectively generating analytic rules extending a basic relational labelled calculus for classical and intuitionistic modal logic. For any (co-)geometric first order formula $\alpha$, the procedure generates a rule $\mathsf{r}$; moreover, if $\alpha$ is the first order correspondent of a modal formula $\varphi$ then $\mathsf{r}$ equivalently captures $\varphi$.

In [2, 10, 11], a class of formulas in the signature of the full Lambek calculus is identified, in the context of a syntactic hierarchy (known as the *proof-theoretic substructural hierarchy*), and an algorithm is introduced for generating analytic rules of a Gentzen-style sequent calculus (resp. hypersequent calculus). This approach was further extended in [3] (generalizing a result for tense modal logic in [9]) to characterize the expressive power of given but not fixed display calculi (from formulas of a given shape to analytic structural rules, and vice versa whenever the calculus satisfies additional conditions).

In [8], a characterization is introduced, analogous to the one of [3] and generalizing [9], of the expressive power of (properly) display calculi, in the context of arbitrary normal (D)LE-logics, i.e. those logics algebraically captured by varieties of normal (distributive) lattice expansions. This characterization is achieved via a systematic connection established between analytic rule-generation and algorithmic correspondence theory [4, 5, 6]. In particular, the same algorithm (ALBA) introduced for generating the first order correspondents of inductive (D)LE-inequalities is used in [8] for generating analytic structural rules of proper display calculi, and the syntactic class of *analytic inductive (D)LE-inequalities* is characterized as those giving rise to properly displayable axiomatic extensions of the basic normal (D)LE-logics.

The contribution discussed in the present talk extends the insights about the systematic connection between algorithmic rule-generation and correspondence theory developed in [8] to relational labelled calculi. Firstly, we use the language of ALBA to encode relational information in a uniform way for any (D)LE-signature; this makes it possible to uniformly design labelled calculi for every basic (D)LE-logic, in which the logical rules encode the behaviour characteristic to each (D)LE-connective in any signature; secondly, we generalize the algorithm MASSA, introduced in [7], to any (D)LE-signature. The general algorithm takes analytic inductive inequalities in input, and outputs (a set of) equivalent analytic rules of a relational labelled calculus. We also show that this algorithm succeeds on all analytic inductive inequalities of any (D)LE-signature.

An important difference between the present algorithmic rule-generation method and Negri's method is that the present method takes *propositional ((D)LE-)inequalities* in input, and, if the input inequality is analytic inductive, it computes its equivalent analytic rule *directly* from the input inequality, via a computation which incorporates the effective generation of its first-order correspondent, whereas Negri's method starts from geometric implications in the first-order frame correspondence language, and generates rules which are equivalent to those modal formulas which are assumed to have a first-order correspondent which is (logically equivalent to) a geometric implication.

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# Universality of the self indexing of a finitely complete category and of its monoidal generalisation

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For some applications of category theory in mathematics and computer science, it is useful for families of objects and morphisms of a category to be indexed not by sets but by set-like objects. Regarding the ability to reindex families as their essential characteristic, one obtains the notion of indexed categories (equivalently, fibrations). An *indexed category* $(\mathbf{S}, \mathbf{C})$ consists of

- a category $\mathbf{S}$—the *index category*—with a terminal object 1,
- for each object $J$ of $\mathbf{S}$, a category $\mathbf{C}^J$ whose objects and morphisms are thought of as the $J$-indexed families of objects and morphisms of the *underlying category* $\mathbf{C}^1$,
- for each morphism $x\colon J \to K$ in $\mathbf{S}$, a functor $\Delta_x\colon \mathbf{C}^K \to \mathbf{C}^J$ which says how to reindex the $K$-indexed families along $x$,

that together form a pseudofunctor $\mathbf{S}^{\mathrm{op}} \to \mathsf{Cat}$.

An *indexing* of a category $\mathbf{C}$ is an indexed category whose underlying category is isomorphic to $\mathbf{C}$. When $\mathbf{C}$ is finitely complete, there is an indexing of $\mathbf{C}$ whose index category is also $\mathbf{C}$, whose category of $J$-indexed families is the slice category $\mathbf{C}/J$, and whose reindexing functors are given by chosen pullbacks; this is the *self indexing* of $\mathbf{C}$. The self indexing of $\mathbf{C}$ provides the foundation for categories internal to $\mathbf{C}$ [5, Section 15], polynomial functors in $\mathbf{C}$ [7], dependent lenses in $\mathbf{C}$ [6], and models of dependent type theory in $\mathbf{C}$. Its ubiquity suggests that it is, at least informally, the canonical indexing of $\mathbf{C}$.

Less well known is that the self indexing of a finitely complete category has a monoidal generalisation. When a symmetric monoidal category $\mathcal{V}$ has well-behaved coreflexive equalisers (i.e. has coreflexive equalisers and these are preserved by the monoidal product in each variable), there is an indexing of $\mathcal{V}$ whose index category is the category of cocommutative comonoids in $\mathcal{V}$ and whose category of $J$-indexed families is the category of $J$-comodules in $\mathcal{V}$; this is the *comonoid indexing* of $\mathcal{V}$ [for $\mathcal{V} = \mathsf{Vect}$, see 2]. If $\mathcal{V}$ is cartesian monoidal, then $\mathcal{V}$ having well-behaved coreflexive equalisers is equivalent to $\mathcal{V}$ being finitely complete; in this case, the comonoid indexing of $\mathcal{V}$ is canonically isomorphic to the self indexing of $\mathcal{V}$. The comonoid indexing of $\mathcal{V}$ gives rise to a notion of category internal to $\mathcal{V}$ that generalises the usual notion of internal category [1]; further investigation into connections with linear dependent lenses and models of linear dependent type theory is warranted. It is conceivable that there could be other indexings of nice monoidal categories that also specialise to the self indexing when the monoidal product is cartesian; to justify calling the comonoid indexing of $\mathcal{V}$ the canonical indexing of $\mathcal{V}$, we need to formalise the notion of canonicity.

Universal properties are one way to formalise notions of canonicity. The functor that sends a finitely complete category to its self indexing is right adjoint to the functor that sends a finitely-complete extensive indexed category to its underlying category; this is closely related to Moens' [4] characterisation of the $\mathbf{S}$-indexed categories that arise from finite-limit-preserving functors $F\colon \mathbf{S} \to \mathbf{C}$ [See 3, Theorem B1.4.12]. In particular, the self indexing of a finitely complete category is terminal amongst the extensive indexings of that category. Significant progress has been made towards proving a similar relationship between underlying categories and comonoid indexings in the indexed monoidal category setting.

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# Elementary fibrations and groupoids

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**Abstract**

The aim of the talk is to illustrate a recent result about the comonadicity of elementary fibrations, and to explain its connections with logical equality.

Lawvere's hyperdoctrines [5, 6] mark the beginning of applications of category theory in logic, and they provide a very clear algebraic tool to work with syntactic theories and their extensions in logic. A ***doctrine*** [7] consists of a family of posets indexed on a category with finite products. More precisely, it is a functor $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{P}os$ into the category $\mathcal{P}os$ of posets, such that the base category $\mathcal{C}$ has finite products. The controvariant action $P(f)\colon P(Y) \to P(X)$ induced by an arrow $f\colon X \to Y$ is called ***reindexing along*** $f$. A (possibly multi-sorted) logical theory $T$ gives rise to a primary doctrine $\mathsf{P}_T$ as follows. The base category consists contexts and context morphisms, *i.e.* finite lists of typed variables and finite lists of typed terms. Composition is given by substitution of terms in terms and product is concatenation of contexts. The poset $\mathsf{P}_T(x_1\colon X_1,\ldots,x_n\colon X_n)$ is the Lindenbaum-Tarksi algebra of formulas in context $(x_1\colon X_1,\ldots,x_n\colon X_n)$. Reindexing along a context morphism $(t_1,\ldots,t_n)\colon (x_1\colon X_1,\ldots,x_m\colon X_m) \to (y_1\colon Y_1,\ldots,y_n\colon Y_n)$ is given by substitution of terms in formulas:

$$\phi \in \mathsf{P}_T(y_1\colon Y_1,\ldots,y_n\colon Y_n) \longmapsto \phi[{}^{t_1}/_{y_1},\ldots,{}^{t_n}/_{y_n}] \in \mathsf{P}_T(x_1\colon X_1,\ldots,x_m\colon X_m).$$

Extending the theory amounts to equip the doctrine with additional structure which, in the spirit of functorial semantics, it is done by requiring certain structural functors to be adjoints. For example, theories with conjunctions correspond to those doctrines, called ***primary***, whose fibres have binary meets which are preserved by reindexing. Adding equality predicates amounts to require that every reindexing along a diagonal $\mathrm{pr}_{1,2,2}\colon Z \times X \to Z \times X \times X$ is right adjoint and satisfies two technical conditions, known as Frobenius Reciprocity and the Beck–Chevalley Condition. These are known as ***elementary doctrines***.

Morphisms between doctrines can be understood as interpretations of a theory into another one, and these can be equipped with a notion of morphism too, giving rise to a 2-category **Doc**. Consider for instance the power set doctrine $\mathcal{P}$ on $\mathcal{S}et$, whose fibre over a set $S$ is the poset of subsets, and reindexing is given by counter-image. Then morphisms into $\mathcal{P}$ are precisely models à la Tarski and, when the theory in question is based on classical first order logic, morphisms between such models are elementary embeddings. Clearly, the model will soundly interpret some logical constant if and only if the morphism into $\mathcal{P}$ preserves the corresponding structure.

The algebraic character of the theory of doctrines makes it a suitable context where to address the question: "What is the theory obtained by (co)freely adding logical structure?", or the closely related question: "How to express additional logical structure in terms of what is already available?". More precisely, in the first case we ask whether a certain forgetful functor is adjoint and, in the second case, whether the adjunction obtained in this way is (co)monadic.

*Speaker.

As a case in point, the forgetful functor from from elementary doctrines to primary ones is comonadic, meaning that elementary doctrines are equivalent to coalgebras for a certain 2-comonad $\mathsf{C}$ on **PD** [2]. It is also known that elementary doctrines with quotients are (pseudo) monadic over elementary ones [8] and, interestingly enough, the 2-monad $\mathsf{M}$ canonically induced on **ED** turns out to be the one presenting elementary doctrines with quotients as (pseudo) algebras. In particular, the diagram below can be recovered just from the 2-comonad $\mathsf{C}$.

$$\begin{tikzcd}
\mathbf{QED} \arrow[r, hook, shift left] \arrow[dr, "\sim"] & \mathbf{ED} \arrow[l, shift left] \arrow[loop above, "\mathsf{M}"] \arrow[r, hook, shift right] \arrow[d, shift right] \arrow[dr, "\sim"] & \mathbf{PD} \arrow[l, shift right] \arrow[loop right, "\mathsf{C}"] \arrow[d, shift right] \\
 & \mathbf{Ps}\text{-}\mathsf{M}\text{-}\mathbf{Alg} \arrow[u, hook, shift right] & \mathsf{C}\text{-}\mathbf{Coalg} \arrow[u, hook, shift right]
\end{tikzcd}$$

As much as doctrines are well suited to deal with standard “proof-irrelevant” logical systems, they fail to capture the additional complexity present in systems such as type theories, where one wishes to keep track of the different proofs of an entailment $\phi \vdash \psi$. To this aim, it is natural to look at indexed categories instead of indexed posets. These are equivalent, via the Grothendieck construction, to Grothendieck fibrations, which have a more robust theory. Under this equivalence, indexed posets are recovered as those fibrations whose underlying functor is faithful. The description of additional logical structure remains the same as in the faithful case since it is given by adjunctions.

After reviewing the above background and motivation, I will explain how to generalise the comonadicity result to the case of Grothendieck fibrations. An interesting byproduct of the comonadicity of elementary fibrations is that the construction of the 2-comonad is finitary and it shows what the logical intuition supported by the case of doctrines evolves to in the general case: it involves in a crucial way the notion of groupoid.

In fact, the proof is based on a characterisation of elementary fibrations that expose similarities with the structures needed to soundly interpret Martin-Löf’s identity type [3]. An instance of such similarities can be found in the fact that Hofmann and Streicher’s interpretation of Martin-Löf’s identity type in groupoids [4] can be presented as an elementary fibration.

As natural as it is to consider groupoids as higher analogues of equivalence relation, other choices are possible, which I will briefly discuss. For example, the homotopy exact completion of a path category [1] is a coalgebra in the 2-category of fibrations with products. Finally, and if times allow, I plan to discuss work in progress to lift the above diagram to fibrations.

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# Some Topological Considerations on Orthogonality

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**Introduction.** This text explores some open lines of investigation related to a class of birelational structures called *orthogonal frames* [2, 1, 4]; these are relational structures $(X, \equiv_1, \equiv_2)$ provided with two equivalence relations which are *orthogonal* to each other, in the sense that $\equiv_1 \cap \equiv_2 = Id_X$.

The logic of orthogonal frames is the fusion $S5 \oplus S5$ [4].

Orthogonal frames are rather ubiquitous in the Modal Logic literature; among other things, they generalise products of Kripke frames [5]. This abstract delves deeper into a off-hand remark made in [4]: namely, the fact that both *subset spaces* and *topological spaces* can be 'seen as' orthogonal frames (in the sense that they are categorically equivalent to a certain class of these frames).

**Definition 1.** Recall that a *subset space* $(X, \tau)$ consists of a nonempty set $X$ and a nonempty collection $\tau$ of subsets of $X$. A *topological space* is a subset space where $\varnothing, X \in \tau$, and $\tau$ is closed under arbitrary unions and finite intersections.

**An orthogonal frame from a topological space.**

**Definition 2.** Given a subset (or topological) space $(X, \tau)$, we construct its *associated orthogonal frame* $(\mathcal{O}, \equiv, \sim)$ as follows:
$\mathcal{O} = \{(x, U) : x \in U \in \tau\}$;
$(x, U) \equiv (y, V)$ iff $x = y$;
$(x, U) \sim (y, V)$ iff $U = V$.

The reader may check that the above frame is orthogonal, for if $(x, U) \equiv (y, V)$ and $(x, U) \sim (y, V)$, then $(x, U) = (y, V)$. A semantics for subset spaces is discussed in [3], where sentences in a bimodal language containing operators $\Box$ and $K$ are evaluated with respect to pairs $(x, U)$ such that $x \in U \in \tau$, as follows: $x, U \models \Box\phi$ iff $x, V \models \phi$ for all $V \in \tau \cap \mathcal{P}(U)$ with $x \in V$; $x, U \models K\phi$ iff $y, U \models \phi$ for all $y \in U$. If we define a partial order $\geq$ on the above frame $\mathcal{O}$ as follows:

$(x, U) \geq (y, V)$ iff $x = y$ and $U \supseteq V$
(iff $(x, U) \equiv (y, V)$ and $(x, U)(\sim \circ \equiv)(z, W)$ for all $(z, W) \sim (y, V)$),

then the usual relational semantics on the frame $(\mathcal{O}, \geq, \sim)$ coincides with the semantics outlined above.

Given that each subset or topological space has an orthogonal frame associated to it, characterising the exact class of such frames which are 'associated' to one of these spaces becomes the next natural question.

**A topological space from an orthogonal frame.**

**Definition 3.** An *orthogonal subset frame* is a frame $(\mathcal{O}, \equiv, \sim)$ where $\equiv$ and $\sim$ are equivalence relations satisfying:

(1) $\equiv \cap \sim = Id_{\mathcal{O}}$;

(2) if $a'(\equiv \circ \sim)b'$ and $b'(\equiv \circ \sim)a'$ for all $a' \sim a$ and for all $b' \sim b$, then $a \sim b$.

An *orthogonal topological frame* is an orthogonal subset frame which moreover satisfies:

(3) if $a \equiv b$, then there exists some $c$ such that, for all $c' \sim c$, $c'(\equiv \circ \sim)a$ and $c'(\equiv \circ \sim)b$;

(4) for all nonempty $A \subseteq \mathcal{O}$, closed under $\sim$, there is some $b$ such that

(4.1) $\forall a \in A$: $a(\equiv \circ \sim)b$; (4.2) $\forall b' \sim b\ \exists a' \in A$: $a' \equiv b'$.

The following holds:

**Proposition 4.** *An orthogonal subset (resp. topological) frame is isomorphic to the associated orthogonal frame of some unique-up-to-isomorphism subset (resp. topological) space.*

The corresponding space is $(X_{\mathcal{O}}, \tau_O)$, where $X_{\mathcal{O}}$ is the quotient set $\mathcal{O}/_{\equiv}$, and $\tau_\pi = \{\varnothing\} \cup \{U_\pi : \pi \in \mathcal{O}/_{\sim}\}$, where we define $U_\pi := \{\sigma \in X_{\mathcal{O}} : \sigma \cap \pi \neq \varnothing\}$. We note that, by orthogonality, $\sigma \in U_\pi$ if and only if $\sigma \cap \pi$ is a singleton, which provides us a natural way to construct the isomorphism alluded to in Prop. 4.

**Theorem 5.** *The category of orthogonal subset (resp. topological) frames is equivalent to the category of subset (resp. topological) spaces.*

**Relation to point-free topology.** In the *point-free topology* literature (e.g. [6]), a *frame* is a complete lattice $(L, \leq)$ such that, for all $A \subseteq L$ and $b \in X$, $(\bigvee A) \wedge b = \bigvee_{a \in A}(a \wedge b)$. In our orthogonal topological frames, the quotient set $\mathcal{O}/_{\sim}$ constitutes such a lattice (minus its minimum) along with the partial order: $[a]_\sim \preceq [b]_\sim$ iff $a(\leq \circ \sim)b$.

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# On presheaf submonads of quantale enriched categories

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Following Lawvere's point of view that it is worth to regard metric spaces as categories enriched in the extended real half-line $[0,\infty]_+$ (see [1]), we regard both the formal ball monad and the monad that identifies Cauchy complete spaces as its algebras – which we call here the *Lawvere monad* – as submonads of the presheaf monad on the category **Met** of $[0,\infty]_+$-enriched categories. This leads us to the study of general presheaf submonads on $V$-**Cat**, the category of $V$-enriched categories, for a given a quantale $V$, that is, a complete lattice endowed with a symmetric tensor product $\otimes$, with unit $k \neq \bot$, commuting with joins, so that it has a right adjoint hom; this means that, for $u, v, w \in V$, $u \otimes v \le w \;\Leftrightarrow\; v \le \hom(u,w)$. As a category, $V$ is a complete and cocomplete (thin) symmetric monoidal closed category.

The following well-known result (see, [2, Theorem 2.5]) plays a fundamental role in the definiton of the presheaf monad on $V$-**Cat**:

**Theorem.** *For $V$-categories $(X,a)$ and $(Y,b)$, and a $V$-relation $\varphi\colon X {\to\!\!\!\!\!+\;} Y$, the following conditions are equivalent:*

(i) $\varphi\colon (X,a) {-\!\!\!\circ\!\!\!\to} (Y,b)$ *is a $V$-distributor;*

(ii) $\varphi\colon (X,a)^{\mathrm{op}} \otimes (Y,b) \to (V,\hom)$ *is a $V$-functor.*

In particular, the $V$-categorical structure $a$ of $(X,a)$ is a $V$-distributor $a\colon (X,a) {-\!\!\!\circ\!\!\!\to} (X,a)$, and therefore a $V$-functor $a\colon (X,a)^{\mathrm{op}} \otimes (X,a) \to (V,\hom)$, which induces, via the closed monoidal structure of $V$-**Cat**, the *Yoneda $V$-functor* $y_X\colon (X,a) \to (V,\hom)^{(X,a)^{\mathrm{op}}}$. Thanks to the theorem above, $V^{X^{\mathrm{op}}}$ can be equivalently described as $PX := \{\varphi\colon X {-\!\!\!\circ\!\!\!\to} E \mid \varphi\ V\text{-distributor}\}$. Then the structure $\widetilde{a}$ on $PX$ is given by $\widetilde{a}(\varphi,\psi) = [\![\varphi,\psi]\!] = \bigwedge_{x\in X} \hom(\varphi(x),\psi(x))$, for every $\varphi,\psi\colon X {-\!\!\!\circ\!\!\!\to} E$, where by $\varphi(x)$ we mean $\varphi(x,*)$, or, equivalently, we consider the associated $V$-functor $\varphi\colon X \to V$. The Yoneda functor $y_X\colon X \to PX$ assigns to each $x \in X$ the $V$-distributor $x^*\colon X {-\!\!\!\circ\!\!\!\to} E$, where we identify again $x \in X$ with the $V$-functor $x\colon E \to X$ assigning $x$ to the (unique) element of $E$. Then, for every $\varphi \in PX$ and $x \in X$, we have that $[\![y_X(x),\varphi]\!] = \varphi(x)$, as expected. In particular $y_X$ is a fully faithful $V$-functor, being injective on objects (i.e. an injective map) when $X$ is a separated $V$-category. We point out that $(V,\hom)$ is separated, and so is $PX$ for every $V$-category $X$. The assignment $X \mapsto PX$ defines a functor $P\colon V\text{-}\mathbf{Cat} \to V\text{-}\mathbf{Cat}$: for each $V$-functor $f\colon X \to Y$, $Pf\colon PX \to PY$ assigns to each $V$-distributor $X \xrightarrow{\varphi} E$ the distributor $Y \xrightarrow{f^*} X \xrightarrow{\varphi} E$. It is easily checked that the Yoneda functors $(y_X\colon X \to PX)_X$

*Speaker.

define a natural transformation $y\colon 1 \to P$. Moreover, since, for every $V$-functor $f$, the adjunction $f_* \dashv f^*$ yields an adjunction $Pf = (\ ) \cdot f^* \dashv (\ ) \cdot f_* =: Qf$, $Py_X$ has a right adjoint, which we denote by $m_X : PPX \to PX$. It is straightforward to check that $\mathbb{P} = (P, m, y)$ is a 2-monad on $V$-**Cat** – the *presheaf monad* –, which, by construction of $m_X$ as the right adjoint to $Py_X$, is lax idempotent (see [3] for details).

We expand on known general characterisations of presheaf submonads and their algebras, and introduce a new ingredient – conditions of Beck-Chevalley type – which allows us to identify properties of functors and natural transformations, and, most importantly, contribute to a new facet of the behaviour of presheaf submonads. In order to do that, we will introduce the basic concepts needed to the study of $V$-categories and present a characterisation of the submonads of the presheaf monad using admissible classes of $V$-distributors which is based on [4]. Then we introduce the Beck-Chevalley conditions (BC*) which resemble those discussed in [5], with $V$-distributors playing the role of $V$-relations. In particular we show that lax idempotency of a monad $\mathbb{T}$ on $V$-**Cat** can be identified via a BC* condition, and that the presheaf monad satisfies fully BC*. This leads to the use of BC* to present a new characterisation of presheaf submonads.

In the remainder of the talk we will focus on the category $(V\text{-}\mathbf{Cat})^{\mathbb{T}}$ of (Eilenberg-Moore) $\mathbb{T}$-algebras, for submonads $\mathbb{T}$ of $\mathbb{P}$. We will start by reviewing some well-known results and we will conclude by presenting a new characterisation for the $\mathbb{B}$-algebras, where $\mathbb{B}$ is the *formal ball monad* on $V$-**Cat**, a natural generalisation of the formal ball monad on the category of (quasi-)metric spaces (cf. [6, 7]), which is constructed using the spaces of formal balls: the collections of all pairs $(x, r)$, where $x \in X$ and $r \in [0, \infty[$, for each (quasi-)metric space $X$.

This talk is based on joint work with Maria Manuel Clementino. A preprint is available [8].

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# Sahlqvist correspondence for deductive systems

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In this talk we present a Sahlqvist Correspondence Theorem [11] for finitary protoalgebraic logics. Our proof is based on an extension of Sahlqvist theory to various fragments of IPC. A formula in the language

$$\mathcal{L} \quad ::= \quad x \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \neg\varphi \mid 0 \mid 1$$

is said to be

(i) a *Sahlqvist antecedent* if it is constructed from variables, negative formulas, and the constants $0$ and $1$ using only $\wedge$ and $\vee$;

(ii) a *Sahlqvist implication* if either it is positive, or it has the form $\neg\varphi$ for a Sahlqvist antecedent $\varphi$, or it has the form $\varphi \to \psi$ for a Sahlqvist antecedent $\varphi$ and a positive formula $\psi$.

Lastly, a *Sahlqvist quasiequation* is a universal sentence of the form

$$\forall \vec{x}, y, z((\varphi_1(\vec{x}) \wedge y \leqslant z \mathbin{\&} \dots \mathbin{\&} \varphi_n(\vec{x}) \wedge y \leqslant z) \Longrightarrow y \leqslant z),$$

where $y, z$ are distinct variables that do not occur in $\varphi_1, \dots, \varphi_n$ and each $\varphi_i$ is constructed from Sahlqvist implications using only $\wedge$ and $\vee$.

*Remark* 1. The focus on quasiequations (as opposed to formulas or equations) is necessary as we deal with fragments where equations have a very limited expressive power. ⊠

Let PSL, (b)ISL, PDL, IL, and HA be, respectively, the varieties of pseudocomplemented semilattices, (bounded) implicative semilattices, pseudocomplemented distributive lattices, implicative lattices, and Heyting algebras. Furthermore, given a poset $\mathbb{X}$, let $\mathsf{Up}(\mathbb{X})$ be the Heyting algebra of its upsets.

**Theorem 2.** *The following holds for every variety* K *between* PSL, (b)ISL, PDL, IL, *and* HA *and every Sahlqvist quasiequation* $\Phi$ *in the language of* K:

(i) Canonicity*: For every* $\boldsymbol{A} \in \mathsf{K}$*, if* $\boldsymbol{A}$ *validates* $\Phi$*, then also* $\mathsf{Up}(\boldsymbol{A}_*)$ *validates* $\Phi$*, where* $\boldsymbol{A}_*$ *is the poset of the meet irreducible filters of* $\boldsymbol{A}$*;*

(ii) Correspondence*: There exists an effectively computable sentence* $\mathsf{fo}(\Phi)$ *in the language of posets such that* $\mathsf{Up}(\mathbb{X}) \vDash \Phi$ *iff* $\mathbb{X} \vDash \mathsf{fo}(\Phi)$*, for every poset* $\mathbb{X}$*.*

To prove Theorem 2, first we extend Sahlqvist Theorem to IPC using Gödel translation of IPC into S4 [7] and its duality theoretic interpretation (see, e.g., [3]). Then, we develop a discrete duality for each variety K as above (cf. [1]) and utilize it to extend Sahlqvist Theorem to the corresponding fragment of IPC.

*Speaker.

A *logic* $\vdash$ is a finitary substitution invariant consequence relation on the set of formulas of some language. Let $\vdash$ be a logic and $\boldsymbol{A}$ an algebra. A subset $F$ of $\boldsymbol{A}$ is said to be a *deductive filter* of $\vdash$ on $\boldsymbol{A}$ if it is closed under the interpretation of the rules valid in $\vdash$. When ordered under the inclusion relation, the set of deductive filters of $\vdash$ on $\boldsymbol{A}$ forms an algebraic lattice $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ with semilattice of compact elements $\mathsf{Fi}^{\omega}_{\vdash}(\boldsymbol{A})$. Lastly, the poset of meet irreducible elements of $\mathsf{Fi}_{\vdash}(\boldsymbol{A})$ will be denoted by $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$.

In order to extend Sahlqvist Correspondence to arbitrary logics, recall that a logic $\vdash$ is said to have

(i) The *inconsistency lemma* (IL) [10] if for every $n \in \mathbb{Z}^+$ there is a finite set of formulas $\sim_n(x_1, \dots, x_n)$ such that for every set of formulas $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$,

$$\Gamma \cup \{\varphi_1, \dots, \varphi_n\} \text{ is inconsistent iff } \Gamma \vdash \sim_n(\varphi_1, \dots, \varphi_n);$$

(ii) The *deduction theorem* (DT) [2] if for every $n, m \in \mathbb{Z}^+$ there is a finite set $(x_1, \dots, x_n) \Rightarrow_{nm} (y_1, \dots, y_m)$[1] of formulas such that for every set of formulas $\Gamma \cup \{\psi_1, \dots, \psi_n, \varphi_1, \dots, \varphi_m\}$,

$$\Gamma, \psi_1, \dots, \psi_n \vdash \varphi_1, \dots, \varphi_m \text{ iff } \Gamma \vdash (\psi_1, \dots, \psi_n) \Rightarrow_{nm} (\varphi_1, \dots, \varphi_m);$$

(iii) The *proof by cases* (PC) [4, 5] if for every $n, m \in \mathbb{Z}^+$ there is a finite set of formulas $(x_1, \dots, x_n) \curlyvee_{nm}(y_1, \dots, y_m)$ such that for every set of formulas $\Gamma \cup \{\psi_1, \dots, \psi_n, \varphi_1, \dots, \varphi_m, \gamma\}$,

$$\Gamma, \psi_1, \dots, \psi_n \vdash \gamma \text{ and } \Gamma, \varphi_1, \dots, \varphi_m \vdash \gamma \text{ iff } \Gamma, (\psi_1, \dots, \psi_n) \underset{nm}{\curlyvee} (\varphi_1, \dots, \varphi_m) \vdash \gamma.$$

A formula $\varphi$ in $\mathcal{L}$ is *compatible* with a logic $\vdash$ when

(i) If $0$ (resp. $1$) occurs in $\varphi$, then $\vdash$ has the IL (resp. the IL or the DT);

(ii) If $\neg$ (resp. $\to, \vee$) occurs in $\varphi$, then $\vdash$ has the IL (resp. DT, PC).

In this case, for every $k \in \mathbb{Z}^+$ we associate a finite set $\boldsymbol{\varphi}^k(\vec{x}_1, \dots, \vec{x}_n)$ of formulas of $\vdash$ (where each $\vec{x}_i$ is a sequence of length $k$) with $\varphi$ as follows:

(i) If $\varphi = x_i$, then $\boldsymbol{\varphi}^k := \{\vec{x}_i\}$;

(ii) If $\varphi = \psi \wedge \gamma$, then $\boldsymbol{\varphi}^k := \boldsymbol{\psi}^k \cup \boldsymbol{\gamma}^k$;

(iii) If $\varphi = \neg\psi$, then $\vdash$ has the IL and, therefore, we set $\boldsymbol{\varphi}^k := \sim_m(\gamma_1, \dots, \gamma_m)$ where $\boldsymbol{\psi}^k = \{\gamma_1, \dots, \gamma_m\}$;

(iv) The cases where $\varphi$ has the form $\psi \to \gamma$ or $\psi \vee \gamma$ are handled similarly to the previous one.

By a *Sahlqvist quasiequation for a logic* $\vdash$ we signify a Sahlqvist quasiequation

$$\Phi = \forall \vec{x}, y, z((\varphi_1(x_1, \dots, x_m) \wedge y \leqslant z \,\&\, \dots \,\&\, \varphi_n(x_1, \dots, x_m) \wedge y \leqslant z) \Longrightarrow y \leqslant z),$$

where $\varphi_1, \dots, \varphi_n$ are compatible with $\vdash$. With it, we associate the set $\mathcal{R}(\Phi)$ of metarules for $\vdash$ of the form

$$\frac{\Gamma, \boldsymbol{\varphi}_1^k(\vec{\gamma}_1, \dots, \vec{\gamma}_m) \vdash \psi, \dots, \Gamma, \boldsymbol{\varphi}_n^k(\vec{\gamma}_1, \dots, \vec{\gamma}_m) \vdash \psi}{\Gamma \vdash \psi.}$$

where $k \in \mathbb{Z}^+$, $\Gamma \cup \{\psi\}$ is a set of formulas, and $\vec{\gamma}_1, \dots, \vec{\gamma}_m$ are sequences of formulas of length $k$.

A logic is *protoalgebraic* if there exists a set of formulas $\Delta(x, y)$ such that $\emptyset \vdash \Delta(x, x)$ and $x, \Delta(x, y) \vdash y$. Our general Sahlqvist Correspondence Theorem takes the following form:

[1] We signify that $\Rightarrow_{nm}$ is a set of formulas in the variables $x_1, \dots, x_n, y_1, \dots, y_m$ by the more suggestive notation $(x_1, \dots, x_n) \Rightarrow_{nm} (y_1, \dots, y_m)$. A similar convention applies to Condition (iii).

**Sahlqvist Correspondence.** *Let $\Phi$ be a Sahlqvist quasiequation for a protoalgebraic logic $\vdash$. Then,*

$$\vdash \textit{ validates the metarules in } \mathcal{R}(\Phi) \quad \textit{iff} \quad \mathsf{Spec}_{\vdash}(\boldsymbol{A}) \vDash \mathsf{fo}(\Phi) \textit{ for every algebra } \boldsymbol{A}.$$

As a consequence, we obtain for instance that a protoalgebraic logic with an IL satisfies a generalization of the excluded middle law (resp. of the bounded top width $n$ formula) iff it is semisimple (resp. principal upsets in $\mathsf{Spec}_{\vdash}(\boldsymbol{A})$ have at most $n$ maximal elements, for every algebra $\boldsymbol{A}$) [8, 9]. The results of this talk are collected in [6].

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# The Structure of Totally Ordered Idempotent Residuated Lattices

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Residuated lattices encompass a wide array of prominent algebraic structures, with examples including Boolean algebras, Heyting algebras, MV-algebras, De Morgan monoids, relation algebras, and lattice-ordered groups, among many others. Thanks to their diversity, residuated lattices provide a unified treatment of substructural logics, for which they give the equivalent algebraic semantics, as well as connecting these logics to classical algebra. However, this diversity also presents a challenge to offering a broadly-applicable analysis of their structure. One approach to addressing this challenge centers on residuated lattices whose multiplication operation is idempotent. Such algebras have proven important, on both the algebraic and logical level, as components in decomposition theorems for more general residuated lattices (see, e.g., [8, 4]), and also complement the already extensively-pursued study of cancellative residuated lattices (see, e.g., [2, 1]). Analyzing the structure of broad classes of residuated lattices based on associated idempotent algebras depends on obtaining structural descriptions of idempotent residuated lattices themselves.

This study focuses on the structure of totally ordered idempotent residuated lattices, advancing a line of research represented in, e.g., [7, 6, 3]. The right and left inverse operations $x^r = x\backslash 1$ and $x^\ell = 1/x$, where $\backslash$ and $/$ are the two residuals of the underlying monoid operation, play an important role in our inquiry, and are crucial in our study of congruences and subalgebra generation in idempotent residuated chains. Among other things, the properties of the inverse operations allow us to establish the following.

**Theorem 1.** *The variety of idempotent semilinear residuated lattices has the congruence extension property.*

Inverses also play a pronounced role in the global structure of idempotent residuated chains. In any idempotent residuated chain, the set of elements that are inverses forms a skeleton, which may be realized as the image of a nucleus. We show that it is possible to reconstruct any given totally ordered idempotent residuated lattice as an ordinal sum indexed by its skeleton through considering the partition induced by this nucleus. Further, we characterize the idempotent residuated chains appearing as skeletons by means of a simple identity, which, in the commutative case, identifies the skeletal idempotent residuated chains as odd Sugihara monoids (see [5, 7]).

We further establish that each totally ordered idempotent residuated lattice is determined by its order and inverse operations, together with the multiplicative identity, and illustrate how the multiplication and division operations may be defined from these ingredients. This analysis supports our introduction of *enhanced monoidal preorders*, enrichments of the monoidal preorders considered in [7], and allows us to establish the following result.

*Speaker.

**Theorem 2.** *Totally ordered idempotent residuated lattices are definitionally equivalent to enhanced monoidal preorders.*

Enhanced monoidal preorders, together with a closely-related graphical presentation of the action of inverses that we call *flow diagrams*, prove a powerful tool for solving problems regarding idempotent residuated chains. We deploy this technology to locate properties causing the failure of the amalgamation property for idempotent residuated chains, which is known to hold under the additional assumption of commutativity. Having pinpointed features that cause amalgamation to fail in the general case, we identify a natural class of idempotent residuated chains for which the amalgamation property holds. In particular, the aforementioned analysis reveals the importance of the derived operation given by $x^\star = x^\ell \wedge x^r$ and suggests consideration of the class of $^\star$-*involutive* idempotent residuated chains defined by $x = x^{\star\star}$. Using the structural results mentioned previously together with variants of some results from [10], we establish the following.

**Theorem 3.** *The class of* $^\star$*-involutive idempotent chains has the strong amalgamation property, and consequently so does the variety of* $^\star$*-involutive idempotent semilinear residuated lattices.*

Because the algebras we consider in this inquiry give the algebraic semantics of certain substructural logics, this work fits into the broader study of metalogical properties of non-classical logics (see e.g. [9]). Via the well known bridge theorems of abstract algebraic logic, we obtain as corollaries of our algebraic results several important metalogical properties for the corresponding logics, including the interpolation property, the Beth definability property, and deduction-detachment theorem.

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# Lattice-ordered groups via distributive lattice-ordered monoids

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**Abstract**

We show that the equational theory of lattice ordered groups ($\ell$-groups) reduces to that of distributive lattice-ordered monoids (DLMs) and that DLMs have the finite model property. Furthermore, we provide an axiomatization for the variety of representable DLMs and show that they satisfy fewer equations than the inverse-free reducts of representable $\ell$-groups. We also provide a link to right orders on the free group and the free monoid.

A *lattice-ordered group* (*$\ell$-group*) consists of a lattice and a group on the same set such that multiplication distributes over meet and join. Lattice-ordered groups have a long history and a rich algebraic theory. We denote by $\mathsf{LG}$ the variety of $\ell$-groups.

A *distributive lattice-ordered monoid* (*DLM*) is a lattice and a monoid on the same set such that multiplication distributes over meet and join and lattice distributivity holds. We denote by $\mathsf{DLM}$ the variety of DLMs. It is easy to see that lattice distributivity follows from the $\ell$-group axioms, so the lattice-monoid reducts of $\ell$-groups are DLMs.

An *abelian* $\ell$-group is an $\ell$-group where multiplication is commutative; a commutative DLM is one where multiplication is commutative. We denote by $\mathsf{ALG}$ and $\mathsf{CDLM}$ the corresponding varieties.

Given an $\ell$-group equation $\varepsilon$, by 'clearing denominators' $\varepsilon$ can be transformed to an inverse-free equation $\varepsilon'$ such that $\mathsf{ALG} \models \varepsilon \Leftrightarrow \mathsf{ALG} \models \varepsilon'$. Unfortunately, Repniskii [5] showed that the further equivalence $\mathsf{ALG} \models \varepsilon' \Leftrightarrow \mathsf{CDLM} \models \varepsilon'$ fails (there are inverse-free equations that hold in $\mathsf{ALG}$ but not in $\mathsf{CDLM}$), so we cannot easily reduce the equational theory of $\mathsf{ALG}$ to that of $\mathsf{CDLM}$. Actually, Repniskii provided an infinite axiomatization of the inverse-free equational theory of $\mathsf{ALG}$ relative to that of $\mathsf{CDLM}$ and proved no finite one exists.

We prove [1] that this discrepancy also holds in the representable case. An $\ell$-group (DLM) is called *representable* (or *semilinear*) if it is a subdirect product of totally-ordered $\ell$-groups (DLMs, respectively); the associated varieties are denoted by $\mathsf{RLG}$ and $\mathsf{RDLM}$. In [1] we provide an equational basis for $\mathsf{RDLM}$.

**Theorem 1.** *The equivalence* $\mathsf{RLG} \models \varepsilon \Leftrightarrow \mathsf{RDLM} \models \varepsilon$ *fails, for some inverse-free equation* $\varepsilon$.

It comes as a surprise that even though for both the commutative and the representable case more equations are satisfied by the corresponding inverse-free reducts of $\ell$-groups than by the corresponding DLMs, in the general case this discrepancy does not appear.

**Theorem 2.** *The equivalence* $\mathsf{LG} \models \varepsilon \Leftrightarrow \mathsf{DLM} \models \varepsilon$ *holds, for every inverse-free equation* $\varepsilon$.

As part of the proof of Theorem 2, we prove that the variety $\mathsf{DLM}$ is generated by the DML of order permutations of the chain of the rationals.

Due to the lack of commutativity, clearing denominators is not as obvious in $\mathsf{LG}$ as in $\mathsf{ALG}$, so it is not clear if we can make use of Theorem 2 in order to reduce the equational theory of $\mathsf{LG}$ to that of $\mathsf{DLM}$ by a string of equivalences:

$$\mathsf{LG} \models \varepsilon \Leftrightarrow \mathsf{LG} \models \varepsilon' \Leftrightarrow \mathsf{DLM} \models \varepsilon' \qquad (*)$$

where for every $\ell$-group equation $\varepsilon$, $\varepsilon'$ is an inverse-free equation corresponding to it. It was the second equivalence that failed in the commutative case and the first one that was true; now the second equivalence is true. It is equally surprising that we can also 'clear denominators' in the non-commutative case (this is not possible in groups, but we prove it holds in $\ell$-groups by making crucial use the join operation). The following result was inspired by the proof of density-elimination in proof theory.

**Theorem 3.** *For every $\ell$-group equation $\varepsilon$, there exists an (effectively constructible) inverse-free equation $\varepsilon'$ such that $\mathsf{LG} \models \varepsilon \Leftrightarrow \mathsf{LG} \models \varepsilon'$.*

Therefore, (*) holds and thus we obtain the following result.

**Corollary 4.** *The equational theory of $\mathsf{LG}$ can be effectively reduced to the equational theory of $\mathsf{DLM}$.*

There exists an analytic Gentzen-Dunn-Mints-style sequent calculus for $\mathsf{DLM}$ (so it enjoys cut elimination and has the subformula property), but unfortunately it is not known how to extract a decision procedure from it. However, the following result shows how to obtain decidability of the equational theory of $\mathsf{DLM}$, and ultimately of $\mathsf{LG}$ by invoking (*). This yields an alternative proof of the decidability of $\mathsf{LG}$ (see [4], [2]) that avoids relying on Holland's embedding theorem [3].

**Theorem 5.** *The variety $\mathsf{DLM}$ has the finite model property.*

Finally, we connect our study to the theory of right orders (and even to right preorders). A *right order* on a monoid is a total order that is compatible with right multiplication.

**Theorem 6.** *Every right order on the free monoid over a set $X$ extends to a right order over the free group over $X$.*

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# Combination of Quantifier-Free Uniform Interpolants using Beth Definability (Abridged Version)

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## Combined Uniform Interpolation

We present recent results on *combination of uniform interpolants* [2]. We recall what uniform interpolants are in general. We fix a logic or a theory $T$ and a suitable fragment $L$ (propositional, first-order quantifier-free, etc.) of its language. Given an $L$-formula $\phi(\underline{x}, \underline{y})$ (here $\underline{x}, \underline{y}$ are the variables occurring in $\phi$), a *uniform interpolant (UI)* of $\phi$ (w.r.t. $\underline{y}$) is a formula $\phi'(\underline{x})$ where only the $\underline{x}$ occur, and satisfying the following two properties: *(i)* $\phi(\underline{x}, \underline{y}) \vdash_T \phi'(\underline{x})$; *(ii)* for any further $L$-formula $\psi(\underline{x}, \underline{z})$ such that $\phi(\underline{x}, \underline{y}) \vdash_T \psi(\underline{x}, \underline{z})$, we have $\phi'(\underline{x}) \vdash_T \psi(\underline{x}, \underline{z})$. Whenever existing, a uniform interpolant for an entailment like $\phi(\underline{x}, \underline{y}) \vdash_T \psi(\underline{x}, \underline{z})$ is computed *independently* of $\psi$.

Uniform interpolants were originally studied in non-classical logics, starting from the pioneering work by Pitts [6]. They are a stronger notion than *ordinary Craig interpolants*: indeed, even in the case Craig interpolants exist, uniform interpolants may not exist. Hence, the existence of uniform interpolants is an exceptional phenomenon, but not so infrequent. Since the nineties, they have been extensively studied in a large literature (e.g., [3, 5]).

Recently, the automated reasoning community has developed an increasing interest in uniform interpolants, focusing on the case $L$ is the *quantifier-free* fragment of some first-order theory $T$: from now on, we restrict our attention to this case. This interest is confirmed, e.g., by Gulwani and Musuvathi in [4], where examples of UI computations were supplied and some algorithms were sketched. The usefulness of uniform interpolants in model checking was first stressed in that work, and then further motivated by data-aware process verification [1].

An important question suggested by model checking concerns the UI transfer to combined theories: *supposing that uniform interpolants exist in theories $T_1, T_2$, under which conditions do they exist also in the combined theory $T_1 \cup T_2$?* We show that combined uniform interpolants exist in the disjoint signatures convex case under the same hypothesis (i.e., the equality interpolating condition) guaranteeing the transfer of quantifier-free ordinary interpolation. For convex theories we essentially obtain a necessary and sufficient condition. The equality interpolating condition is not sufficient for the non-convex case (see [2] for a counterexample).

**Main results.** A theory $T$ is *convex* iff for every constraint $\delta$, if $T \vdash \delta \to \bigvee_{i=1}^{n} x_i = y_i$ then $T \vdash \delta \to x_i = y_i$ holds for some $i \in \{1, ..., n\}$. Horn theories are convex, but there exist non-Horn convex theories such as $Th(\mathbb{R}, 0, +, -, =, <)$. We need the following definition:

**Definition 1.** *A convex universal theory $T$ is* equality interpolating *iff for all variables $y_1, y_2$ and for every pair of* constraints *$\delta_1(\underline{x}, \underline{z}_1, y_1), \delta_2(\underline{x}, \underline{z}_2, y_2)$ s.t. $T \vdash \delta_1(\underline{x}, \underline{z}_1, y_1) \wedge \delta_2(\underline{x}, \underline{z}_2, y_2) \to y_1 = y_2$, there is a term $t(\underline{x})$ s.t. $T \vdash \delta_1(\underline{x}, \underline{z}_1, y_1) \wedge \delta_2(\underline{x}, \underline{z}_2, y_2) \to y_1 = t(\underline{x}) \wedge y_2 = t(\underline{x})$.*

*Speaker.

We recall that a universal theory $T$ has quantifier-free interpolation iff $T$ enjoys amalgamation. In case $T$ is also equality interpolating, a stronger characterization holds:

**Fact 1.** *The following are equivalent for a convex universal theory $T$: (i) $T$ is equality interpolating and has quantifier-free interpolation; (ii) $T$ has the strong amalgamation property.*

Consider a primitive formula $\exists \underline{z}\phi(\underline{x}, \underline{z}, y)$: $\exists \underline{z}\, \phi(\underline{x}, \underline{z}, y)$ *implicitly defines* $y$ in $T$ iff the formula $\forall y\, \forall y'\ (\exists \underline{z}\phi(\underline{x}, \underline{z}, y) \wedge \exists \underline{z}\phi(\underline{x}, \underline{z}, y') \to y = y')$ is $T$-valid; $\exists \underline{z}\phi(\underline{x}, \underline{z}, y)$ *explicitly defines* $y$ in $T$ iff there is a term $t(\underline{x})$ s.t. the formula $\forall y\ (\exists \underline{z}\phi(\underline{x}, \underline{z}, y) \to y = t(\underline{x}))$ is $T$-valid. A theory $T$ has the *Beth definability property (for primitive formulae)* iff whenever $\exists \underline{z}\, \phi(\underline{x}, \underline{z}, y)$ implicitly defines the variable $y$ then it also explicitly defines it. It is worth noticing the following result:

**Fact 2.** *A convex equality interpolating theory $T$ has the Beth definability property.*

Let us fix two theories $T_1, T_2$ over disjoint signatures $\Sigma_1, \Sigma_2$, satisfying the assumptions of Theorem 1 below. Our problem is to compute a uniform interpolant for $\phi(\underline{x}, \underline{y})$ (w.r.t. $\underline{y}$), where $\phi$ is a conjunction of $\Sigma_1 \cup \Sigma_2$-literals. In order to design a combined UI algorithm (called ConvexCombCover and shown in detail in [2]), we exploit the equivalence between implicit and explicit definability that is supplied by Beth definability: the algorithm guesses the implicitly definable variables, then eliminates them via explicit definability, and finally uses the component-wise input UI algorithms to eliminate the remaining (not implicitly definable) variables. The identification and the elimination of the implicitly defined variables via explicitly defining terms is essential for the correctness of the combined UI algorithm: when computing a uniform interpolant of $\phi(\underline{x}, \underline{y})$ (w.r.t. $\underline{y}$), the variables $\underline{x}$ are (non-eliminable) parameters, and those variables among the $\underline{y}$ that are implicitly definable *need to be discovered and treated in the same way as the parameters $\underline{x}$*. Only after this, the input UI algorithms can be exploited.

**Theorem 1.** *Let $T_1, T_2$ be convex, stably infinite, equality interpolating, universal theories over disjoint signatures admitting uniform interpolants. Then $T_1 \cup T_2$ admits uniform interpolants too. Uniform interpolants in $T_1 \cup T_2$ can be effectively computed using* ConvexCombCover.

The previous theorem shows that the equality interpolating condition is sufficient for transferring uniform interpolants to combinations. In [2], it is also shown that equality interpolating is a necessary condition for obtaining UI transfer, in the sense that it is already required for minimal combinations with signatures adding uninterpreted symbols.

The combination result we obtain is quite strong, as it is a typical 'black box' combination result: it applies not only to theories used in verification (such as the combination of real arithmetics with uninterpreted functions), but also in other contexts.

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# Preserving joins at primes

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The aim of this talk is to show an instance of interaction between lattice theory, domain theory, and profinite monoids. Our main aim is to present an algebraic concept, *preserving joins at primes*, and develop a duality theory for it. Towards the end, we indicate two different applications of this concept in the foundations of computer science.

**Implication operators.** We call an *implication operator* on a bounded distributive lattice $L$ a binary operation $\Rightarrow\colon L^{\mathrm{op}} \times L \to L$ such that, for any elements $a, a', b, b'$ of $L$,

1. $\bot \Rightarrow a = \top = a \Rightarrow \top$,
2. the following two equalities hold in $L$:

$$
\begin{aligned}
(a \vee a') \Rightarrow b &= (a \Rightarrow b) \wedge (a' \Rightarrow b), \\
a \Rightarrow (b \wedge b') &= (a \Rightarrow b) \wedge (a \Rightarrow b').
\end{aligned}
$$

Examples of implication operators in this sense occur frequently in the algebraic study of non-classical logics: the implication of a Heyting algebra is one example, as is the implication on an MV-algebra. Implication operators in general do not need to preserve disjunctions in the second coordinate; i.e., in logical terms, for formulas $A, B$ and $C$, $(A \Rightarrow B) \vee (A \Rightarrow C)$ is stronger than $A \Rightarrow B \vee C$, and the two are not always equivalent.

**Preserving joins at primes.** We say an implication operator $\Rightarrow$ on a bounded distributive lattice $L$ *preserves joins at primes* if (i) $a \Rightarrow \bot = \bot$ for any $a \in L \setminus \{\bot\}$, and, (ii) for any prime filter $x$ of $L$, for any $a \in x$ and for any $b, c \in L$, there exists $a' \in x$ such that

$$
a \Rightarrow (b \vee c) \leq (a' \Rightarrow b) \vee (a' \Rightarrow c).
$$

To explain the name, although we do not strictly need this in what follows, we note that an equivalent formulation of this notion is the following, using the *canonical extension* $L^\delta$ of the bounded distributive lattice $L$ [4]. An implication operator $\Rightarrow$ on $L$ preserves joins at primes iff for any completely join-prime element $x$ of $L^\delta$, the following function preserves finite joins:

$$
\begin{aligned}
x \Rightarrow (-)\colon L &\to L^\delta, \\
b &\mapsto \bigvee \{a \Rightarrow b \mid x \leq a \in L\}.
\end{aligned}
$$

*Speaker.

**Duality.** Note that an implication operator $\Rightarrow$ on $L$ can be alternatively given by a lattice homomorphism $[\![-]\!]\colon F_{\Rightarrow}(L) \to L$, where $F_{\Rightarrow}(L)$ is the quotient of the free distributive lattice over $L \times L$ by the congruence generated by the equalities defining the notion of implication operator. This construction $L \mapsto F_{\Rightarrow}(L)$ can be made into a functor on distributive lattices. Moreover, by Priestley duality, a lattice homomorphism $[\![-]\!]\colon F_{\Rightarrow}(L) \to L$ corresponds to a continuous order-preserving map $r\colon X_L \to R(X_L)$, where $X_L$ denotes the Priestley dual space of $L$, and $R$ denotes the construction dual to $F_{\Rightarrow}$. Recall that a Priestley space $X$ may alternatively be described via its topology of open upward closed sets, $X^{\uparrow}$, that we call the *spectral space* associated to $X$. We now give an explicit description of the object part of the functor $R$, viewed on spectral spaces. For a spectral space $S$, let us denote by $R(S)$ the *binary relation space* on $S$, i.e., the space of continuous functions from $S$ to the *upper Vietoris space* $\mathcal{V}(S)$ of $S$, equipped with the compact-open topology.

In what follows, let $L$ be a bounded distributive lattice with dual spectral space $S$.

**Theorem 1.** *The dual space of $F_{\Rightarrow}(L)$ is order-homeomorphic to the binary relation space on $S$.*

From this theorem and the functorial point of view on implication operators described above, one may deduce in particular the known result that implication operators $\Rightarrow$ on a bounded distributive lattice $L$ are in one-to-one correspondence with ternary relations on the dual space $X$ that satisfy a number of topological conditions, see e.g. [1]. Building on the above theorem, one may try to similarly characterize the implication operators that preserve joins at primes. Since the definition of preserving joins at primes is not first-order (it refers to prime filters of $L$), the functorial approach we outlined above for general implication operators does not go through directly. However, we do have the following. Let us say for a lattice congruence $\theta$ on $F_{\Rightarrow}(L)$ that $\Rightarrow$ *preserves joins at primes modulo* $\theta$ if, for any $a \in L \setminus \{\bot\}$, $(a \Rightarrow \bot)\theta\bot$, and for any prime filter $x$ of $L$, $a \in x$, and $b, c \in L$, there is $a' \in x$ such that $a \Rightarrow (b \vee c) \leq_{\theta} (a' \Rightarrow b) \vee (a' \Rightarrow c)$. Finally, denote by $[S, S]$ the (not necessarily spectral) subspace of $R(S)$ consisting of the functions $f\colon X \to \mathcal{V}(S)$ such that $f(x)$ is a principal up-set for every $x \in S$.

**Theorem 2.** *The dual of the quotient of $F_{\Rightarrow}(L)$ by a congruence $\theta$ is a subspace of $[S, S]$ if and only if $\Rightarrow$ preserves joins at primes modulo $\theta$.*

A slight generalization of this theorem also allows one to describe subspaces of $[S, T]$, where $S$ and $T$ are two different spectral spaces, in terms of quotients of a lattice $F_{\Rightarrow}(L, M)$ of implications between elements of $L$ and $M$.

**Applications.** In the theory of bifinite domains [3, 2], one may actually prove that $[S, T]$ is always a bifinite domain, if $S$ and $T$ are. Using the above duality results, a proof of this result can be obtained by showing that, in this special setting, there is a smallest congruence, $\theta_j$, on $F_{\Rightarrow}(L, M)$ such that $\Rightarrow$ preserves joins at primes modulo $\theta_j$, and that this quotient is again bifinite. The dual space of the quotient by this smallest congruence $\theta_j$ is then $[S, T]$. Further, equations between domains, e.g. Scott's solution to $X \cong [X, X] \cong X \times X$, may now be analyzed dually by considering the corresponding lattices.

In the theory of regular languages and profinite monoids, where the notion of preserving joins at primes first appeared in [1], it was shown to characterize exactly those residuated families of implication operators on a Boolean algebra that are dual to a continuous binary operation.

**Related work.** Many of the results presented in this abstract have previously appeared in the literature, in particular in domain theory [3, 2] and topological algebra [1]. We believe the presentation and the connection between them, as mediated by the notion of join-preserving at primes, is novel. This abstract is based on parts of the two final chapters of a forthcoming textbook [5] that we are writing on duality theory, with applications to computer science.

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# Projective unification through duality

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In a propositional language, substitutions can be defined as functions mapping variables to formulas. For reasons related to Unification Theory [BS01, Section 2], it is usually considered that such functions are almost everywhere equal to the identity function. According to this point of view, which is the one usually considered within the context of modal logics [BR11, Dzi07, Ghi00], a substitution is a function $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ where $\mathcal{L}_P$ (resp. $\mathcal{L}_Q$) is the set of all formulas with variables in a finite set $P$ (resp. $Q$), and satisfying ($\blacklozenge$) $\sigma(\circ(\varphi_1, \ldots, \varphi_n)) = \circ(\sigma(\varphi_1), \ldots, \sigma(\varphi_n))$ for all $n$-ary connectives $\circ$ of the language and all formulas $\varphi_1, \ldots, \varphi_n \in \mathcal{L}_P$.

A formula $\varphi \in \mathcal{L}_P$ is $\mathbf{L}$-unifiable if $\mathbf{L}$ contains instances of $\varphi$. In that case, any substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ such that $\sigma(\varphi) \in \mathbf{L}$ counts as a $\mathbf{L}$-unifier of $\varphi$. A $\mathbf{L}$-unifiable formula $\varphi \in \mathcal{L}$ is projective if it possesses a projective $\mathbf{L}$-unifier, that is to say a $\mathbf{L}$-unifier $\sigma$ such that $\varphi \vdash_{\mathbf{L}} \sigma(p) \leftrightarrow p$ holds for all $p \in P$. Such unifiers are interesting because they constitute by themselves minimal complete sets of unifiers [BR11, Dzi07, Ghi00]. For this reason, it is of the utmost importance to be able to determine if a given formula is projective. Ghilardi's proof that transitive modal logics such as **K4** and **S4** are finitary is based on projective unifiers [Ghi00]. In [Sło12], Słomczyńska uses projective unifiers to determine the unification type of some implicational fragments of intuitionistic propositional logic.

Now, condition ($\blacklozenge$) may evoke homomorphism properties. Following this observation, Unification Theory was also formalized and studied in an algebraic setting [Ghi97, Sło12]. Indeed, let us consider the Lindenbaum algebra $\mathbf{A}_P$ obtained by taking the quotient of $\mathcal{L}_P$ modulo the relation $\equiv_{\mathbf{L}}$ of $\mathbf{L}$-equivalence. One can associate to a substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ the map $\sigma^{\triangleright} : \mathbf{A}_P \to \mathbf{A}_Q$ by setting $\sigma^{\triangleright}([\varphi]_{\mathbf{L}}) := [\sigma(\varphi)]_{\mathbf{L}}$ for any formula $\varphi \in \mathcal{L}_P$, whose equivalence class modulo $\equiv_{\mathbf{L}}$ is denoted by $[\varphi]_{\mathbf{L}}$. In this perspective, condition ($\blacklozenge$) then truly expresses the homomorphic character of $\sigma^{\triangleright}$. Obviously, this association between substitutions and homomorphisms of Lindenbaum algebras is one-to-one modulo $\simeq_{\mathbf{L}}$: substitutions associated to the same homomorphism are equivalent modulo $\simeq_{\mathbf{L}}$. Then properties of substitutions, such as being a $\mathbf{L}$-unifier of a formula, admit an algebraic counterpart too.

In this work, we combine this correspondence with a more traditional one, provided by *Duality Theory*. For any set $P$ of variables, there is indeed a tight connection between the Lindenbaum algebra $\mathbf{A}_P$ and the canonical frame $\mathfrak{F}_P$ of $\mathbf{L}$ over $P$, determined by the set of all ultrafilters on $\mathbf{A}_P$. Homomorphisms between Lindenbaum algebras are then in correspondence with bounded morphisms between canonical frames. See [BRV01, Chapter 5], [CZ97, Chapter 7] and [Kra99, Chapter 4] for a general introduction to this subject. Duality has already been employed by Ghilardi [Ghi04] to solve unification problems in Heyting algebras. In our work we make substantial use of it to construct a necessary and sufficient condition for $\varphi \in \mathcal{L}_P$ to be projective. Here a central role is played by $\widehat{\varphi}^{\infty} := \bigcap_{n \in \mathbb{N}} \widehat{\Box^n \varphi}$, i.e. the set of all points in $\mathfrak{F}_P$ containing $[\Box^n \varphi]_{\mathbf{L}}$ for all $n \in \mathbb{N}$. Indeed, we prove that $\varphi$ is projective if and only if there exists a bounded morphism $f : \mathfrak{F}_P \to \mathfrak{F}_P$ such that the image of $f$ is contained in $\widehat{\varphi}^{\infty}$, and all elements of $\widehat{\varphi}^{\infty}$ are fixpoints of $f$.

*Speaker.

After establishing this equivalence, we apply it to study the projective – or non-projective – character of the extensions of the logics **K4**, **K5**, and

$$\mathbf{K4}_n := \mathbf{K} + (\Diamond^{n+1}p \to \Diamond^{\leq n}p)$$
$$\mathbf{K4}_n\mathbf{B}_k := \mathbf{K4}_n + (p \to \Box^{\leq k}\Diamond^{\leq k}p)$$

where $n, k \geq 1$. We show that all extensions of $\mathbf{K4}_n\mathbf{B}_k$ are projective, thereby reproving a recent result of Kostrzycka [Kos22]. The extensions of **K4** were studied by Kost [Kos18], who proved that the projective extensions of **K4** are exactly the extensions of the logic

$$\mathbf{K4D1} := \mathbf{K4} + \Box(\Box p \to q) \vee \Box(\Box q \to p).$$

Here we show that all locally tabular[1] extensions of **K4D1** are projective, and that all projective extensions of **K4** are also extensions of **K4D1**. This is obviously weaker than Kost's result, but still covers a decent range of logics. With a simple adaptation of our proof, we also show that all projective extensions of $\mathbf{K4}_n$ are extensions of

$$\mathbf{K4}_n\mathbf{D1}_n := \mathbf{K4}_n + \Box(\Box^{\leq n}p \to q) \vee \Box(\Box^{\leq n}q \to p).$$

Most interestingly, we prove that the projective extensions of **K5** are exactly the extensions of **K45**. In particular, this resolves the open question of whether **K5** is projective. It should also be noted that our proofs are fairly lightweight and concise, as opposed to syntactic methods, which often involve all sorts of technical twists. Of course, this is only a first insight of what duality has to offer. Hopefully these results will raise interest in this line of work, and open promising new directions.

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[1] A logic **L** is *locally tabular* if for all finite sets $P$ of variables, there are only finitely many formulas in $\mathcal{L}_P$ modulo $\equiv_{\mathbf{L}}$.

# A pronilpotent look at maximal subgroups of free profinite monoids

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Profinite monoids are rich and interesting objects which are related with several topics in both algebra and logic (for instance, logic on words). In the early 2000s, Almeida established a connection between symbolic dynamics and free profinite monoids [1, 2, 3]. The work we present [8] aims to further investigate this connection in order to gain new insights into the structure of the maximal subgroups of free profinite monoids. We do this by taking a closer look at the finite nilpotent quotients of these groups, which in practical terms amounts to computing their maximal pronilpotent quotients.

Let us briefly outline the correspondence introduced by Almeida. A language $L \subseteq A^*$ is called *uniformly recurrent* when

1. it is factorial: if $w \in L$ and $u$ is a factor of $w$, then $u \in L$;
2. it is extendable: if $w \in L$, then $awb \in L$ for some $a, b \in A$;
3. within $L$, all words are eventually unavoidable: for all $u \in L$, there exists $n \in \mathbb{N}$ such that $u$ is a factor of every $v \in L$ with $|v| \geq n$.

In his 2007 paper [3], Almeida proved that to each uniformly recurrent language $L \subseteq A^*$ corresponds a maximal subgroup, well-defined up to isomorphism, of the free profinite monoid $\widehat{A^*}$. This group, which is a projective profinite group [10], lies inside $\overline{L} \setminus A^*$, the "infinite part" of the topological closure of $L$ in $\widehat{A^*}$. It is known as the *Schützenberger group* of $L$, and can be thought of as an invariant for $L$ [6]. In some cases, this invariant is well understood: for instance, when $L$ is a Sturmian language, its Schützenberger group must be a free profinite group of rank 2 [5]. But not all cases are so straightforward: the Schützenberger group of the language of the Thue–Morse word is not free, not even relative to some pseudovariety of finite groups [4].

The key for understanding maximal pronilpotent quotients in the case at hand is a special kind of profinite presentations, which characterize projective objects in the category of profinite groups [9]. Generically, these presentations are of the form $\langle A \mid \varphi(a) = a,\ a \in A\rangle$, where $\varphi$ is an idempotent continuous endomorphism of the free profinite group over $A$. In 2013, Almeida and Costa [4] obtained explicit presentations of the above form for Schützenberger groups corresponding to languages defined by *primitive substitutions* (i.e. endomorphisms of $A^*$ whose composition matrices are primitive in the usual sense). An idempotent continuous endomorphism determining such a presentation can be computed using a *return substitution*, an important notion from symbolic dynamics which was introduced by Durand [7].

A *pronilpotent group* (respectively, *pro-p group*) is an inverse limit, in the category of compact groups, of finite discrete nilpotent groups (respectively, $p$-groups). In order to leverage the aforementioned profinite presentations and obtain a description of the maximal pronilpotent quotients, we rely on a number of fundamental results. First and foremost is Tate's famous characterization of projective pro-$p$ groups, which states that they are all free (a precise statement is found e.g. in [11]). Second is the fact that the maximal pronilpotent quotient functor (left adjoint to the inclusion functor from pronilpotent groups to profinite groups) is naturally isomorphic to the product of the maximal pro-$p$ quotient functors, where $p$ ranges over all primes. (Essentially, for the same reason that finite nilpotent groups are isomorphic to the direct product of their Sylow subgroups.) As a result, the maximal pronilpotent quotient of a projective profinite group must be isomorphic to a product of free pro-$p$ groups. In the specific case of the Schützenberger group corresponding to a primitive aperiodic substitution, we show that there is a transparent relationship between the rank of these pro-$p$ factors on the one hand, and the characteristic polynomial of the composition matrix of any return substitution on the other hand. This means that all the information about the pronilpotent quotients of these groups can be neatly packaged into one single polynomial, which moreover can be effectively computed. Using the close relationship between the characteristic polynomial of a primitive substitution and those of its return substitutions (slightly strengthening a result of Durand [7]), we conclude that the characteristic polynomial of the substitution itself still carries some information about the pronilpotent quotients of the Schützenberger group.

In many cases, some features of a profinite group, such as failure of freeness, are witnessed by its pronilpotent quotients. Using this to our advantage, we devise a number of tests (i.e. necessary conditions) for freeness, both relative and absolute, of the Schützenberger groups corresponding to primitive aperiodic substitutions. These tests require little more than a quick look at the characteristic polynomial, either of the substitution itself or of one of its return substitutions. One such test, particularly easy to perform, can be succinctly phrased as follows: for the maximal subgroup corresponding to a primitive aperiodic substitution to be absolutely free, it is necessary that the product of the non-zero eigenvalues of its composition matrix (in other words, its *pseudodeterminant*) be 1 in absolute value. A notable family of substitutions failing this condition consists of primitive aperiodic substitutions of constant length, which includes the Thue–Morse substitution. In particular, it is now clear that such substitutions never produce free maximal subgroups.

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# Syntactic completeness of proper display calculi

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In recent years, research in structural proof theory has focused on *analytic calculi* [14, 3, 10, 2, 18, 19], understood as those calculi supporting a *robust* form of cut elimination, i.e. one which is preserved by adding rules of a specific shape (the analytic rules). Important results on analytic calculi have been obtained in the context of various proof-theoretic formalisms: (classes of) axioms have been identified for which equivalent correspondences with analytic rules have been established algorithmically or semi-algorithmically. This strand of research has been developed in the context of *sequent and labelled calculi* [16, 17, 15, 14], *sequent and hypersequent calculi* [3, 12, 13], and *(proper) display calculi* [11, 4, 10].

In [10], which is the contribution in the line of research described above to which the results discussed in the present talk most directly connect, a characterization, analogous to the one of [4],[1] of the property of being properly displayable[2] is obtained for arbitrary normal (D)LE-logics[3] via a systematic connection between proper display calculi and generalized Sahlqvist correspondence theory (aka unified correspondence [5, 6, 7, 8]). Thanks to this connection, general meta-theoretic results are established for properly displayable (D)LE-logics. In particular, in [10], the properly displayable (D)LE-logics are syntactically characterized as the logics axiomatised by *analytic inductive axioms* (namely axioms of a given syntactic shape, see [10, Definition 51 and 55]); moreover, the same algorithm ALBA which computes the first-order correspondent of (analytic) inductive (D)LE-axioms can be used to effectively compute their corresponding analytic structural rule(s). In [1], following [9], residuated families of unary and binary connectives are studied parametrically in group actions on the coordinates of the relations associated with the connectives.

The *semantic* equivalence between each analytic inductive axiom $\varphi \vdash \psi$ and its corresponding analytic structural rule(s) $R_1, \ldots, R_n$, discussed in [10], is an immediate consequence of the soundness of the rules of ALBA on perfect normal (distributive) lattice expansions. On the *syntactic* side, a description of the derivation, which relies on the proof-theoretic version of Ackermann's Lemma and therefore involves cuts, is presented in [4]. However, an effective procedure was still missing for building *cut-free* derivations of $\varphi \vdash \psi$ in the proper display calculus obtained by adding $R_1, \ldots, R_n$ to the basic proper display calculus D.LE (resp. D.DLE) of the basic normal (D)LE-logic. Such an effective procedure would establish, via syntactic means, that for any properly displayable (D)LE-logic $\mathsf{L}$, the proper display calculus for $\mathsf{L}$—i.e. the calculus obtained by adding the analytic structural rules obtained from the axioms of $\mathsf{L}$ to the basic calculus D.LE (resp. D.DLE)—derives all the theorems (or derivable sequents) of $\mathsf{L}$. This is what we refer to as the *syntactic completeness* of the proper display calculus for $\mathsf{L}$ with respect to any analytic (D)LE-logic $\mathsf{L}$. This syntactic completeness result for all properly displayable logics in arbitrary (D)LE-signatures is the main contribution of the present research. It is perhaps worth emphasizing that we do not just show that any analytic inductive axiom is derivable in its corresponding proper display calculus, but we also provide an algorithm to generate a *cut-free* derivation of a particular shape that we refer to as being *in pre-normal form*.

[1] For a comparison between the characterizations in [4] and in [10], see [10, Section 9].

[2] A display calculus is *proper* if every structural rule is closed under uniform substitution. A logic is *(properly) displayable* if it can be captured by some (proper) display calculus (see [10, Section 2.2]).

[3] Normal (D)LE-logics are those logics algebraically captured by varieties of normal (distributive) lattice expansions, i.e. (distributive) lattices endowed with additional operations that are finitely join-preserving or meet-reversing in each coordinate, or are finitely meet-preserving or join-reversing in each coordinate.

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# Lambek-Grishin Calculus: focusing, display and full polarization

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*Focused sequent calculi* [1, 2, 11] make use of syntactic restrictions on the applicability of inference rules achieving three main goals: (i) the proof search space is considerably reduced without losing completeness, (ii) every cut-free proof comes in a special normal form, (iii) a criterion for defining identity of sequent calculi proofs. Being able to identify or tell apart two proofs has far-reaching consequences.

We introduce a novel focused display calculus **fD.LG** and a *fully polarized* algebraic semantics (see the last paragraph of this abstract for more details on this) $\mathbb{FP}.\mathbb{LG}$ for Lambek-Grishin logic [12] by generalising the theory of *multi-type calculi* [5] and their algebraic semantics, admitting not only heterogeneous operators [4], but also *heterogeneous consequence relations* (see [9]) now interpreted as *weakening relations* [10] (i.e. a natural generalisation of partial orders). The calculus **fD.LG** has *strong focalization* and it is *sound and complete* w.r.t. $\mathbb{FP}.\mathbb{LG}$. This completeness result is in a sense stronger than completeness with respect to standard polarized algebraic semantics, insofar we do not need to quotient over proofs with consecutive applications of *shifts operator* over the same formula (see the last paragraph of this abstract for more details on this). We also show a number of additional results. **fD.LG** is sound and complete w.r.t. LG-algebras: this amounts to a semantic proof of the *completeness of focusing*, given that the standard (display) sequent calculus for Lambek-Grishin logic is complete w.r.t. LG-algebras. **fD.LG** and the focused calculus **fLG** of Moortgat and Moot are equivalent with respect to proofs, indeed there is an effective translation from **fLG**-derivations to **fD.LG**-derivations and vice versa: this provides the link with operational semantics, given that every **fLG**-derivation is in a Curry-Howard correspondence with a directional $\overline{\lambda}\mu\widetilde{\mu}$-term.

We conjecture that this approach, here tailored for the signature of the Lambek-Grishin logic, can be extended to a large class of logics, namely all lattice expansions logics extended with *analytic inductive axioms* (see [6]). We conjecture that if a calculus belongs to this class, then it enjoys cut-elimination, aiming at generalizing the cut-elimination meta-theorem in the tradition of display calculi (see [13]). Moreover, we conjecture that any *displayable logic* [6] can be equivalently presented as an instance of this class.

In what follows we summarise the main features of this analysis in general terms, without special reference to Lambek-Grishin logic. In the case of focused sequent calculi, the distinction between *positive* versus *negative* formulas is the key ingredient for organising proofs in *phases.* The distinction is proof-theoretically relevant in that it reflects a fundamental distinction between logical introduction rules. We observe that this distinction is also semantically

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grounded, indeed the main connective of a positive formula (in the original language of the logic) is a left adjoint/residual and the main connective of a negative formula (in the original language of the logic) is a right adjoint/residual. Proofs in *focalized normal form* (see [12]) are cut-free proofs organised in three phases: two focused phases (either positive or negative) and one non-focused phase (also called neutral phase). A focused positive (resp. negative) phase in a derivation is a proof-section where a formula is decomposed as much as possible only by means of non-invertible logical rules for positive (resp. negative) connectives. This formula and all its immediate subformulas in this proof-section are said 'in focus'. All the other rules are applied only in non-focused phases. So, each derivable sequent has at most one formula in focus. Moreover, the interaction between two focused phases is always mediated by a non-focused phase.

*Shift operators* –usually denoted as $\uparrow$ and $\downarrow$ ([7, 8, 3])– are often considered to polarize a focused sequent calculus, i.e. as a tool to control the interplay between positive and negative formulas and the interaction between phases. Shifts are adjoint unary operators that change the polarity of a formula, where $\uparrow$ goes from positive to negative, $\downarrow$ goes from negative to positive, and $\uparrow \dashv \downarrow$. In this paper, we consider positive and negative formulas as formulas of different sorts. We also distinguish between positive (resp. negative) *pure* formulas and positive (resp. negative) *shifted* formulas, i.e. formulas under the scope of a shift operator. So, we end up considering four different sorts, each of which is interpreted in a different sub-algebra. Therefore, in this setting shifts are heterogeneous operators, where $\uparrow$ gets split into $\uparrow$ (from positive pure formulas into negative shifted formulas) and $\upharpoonleft$ (from positive shifted formulas into negative pure formulas), $\downarrow$ gets split into $\downarrow$ (from negative pure formulas into positive shifted formulas) and $\downharpoonright$ (from negative shifted formulas into positive pure formulas), $\uparrow \dashv \downharpoonright$ and $\upharpoonleft \dashv \downarrow$. Moreover, the composition of two shifts is still either a closure or an interior operator (by adjunction), but we do not assume that it is an identity. We call a presentation of a logic with the features described above *full* polarization.

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# Łukasiewicz logic properly displayed

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*Mathematical fuzzy logics* [4] are often motivated by semantic considerations, namely representing and reasoning about truth degrees. Hilbert systems are a convenient formalism for presenting logics corresponding to classes of algebras and they were abundantly used in presenting and organising various mathematical fuzzy logics, many of which come in large subclasses with specific properties [8].

*Structural proof theory* [15] studies the structure and properties of proofs and in this context sequent calculi are a fundamental tool in showing that proofs can be organised as to preserve *analyticity*. A core line of research (see for instance [14, 2, 3, 11, 10, 5]) focuses on the *algorithmic generation* of *analytic rules*, namely rules that preserve the analyticity whenever added to an analytic calculus.

Łukasiewicz logic is one of the most well-known and thoroughly studied mathematical fuzzy logics (see [13] for an overview of proof theoretic literature on mathematical fuzzy logics), and various (analytic) calculi capturing this logic were proposed and studied: for instance, [12] introduces various sequent-style calculi (hypersequent calculi, labelled sequent calculi and unlabelled sequent calculi) for the $\{\mathbf{0}, \rightarrow\}$-fragment of Łukasiewicz (and Abelian logic), while [1] introduces so-called relational hypersequent calculi for the full fragment of Łukasiewicz but $\{\mathbf{1}, \ominus\}$ (and Product and Gödel logics as well). Nonetheless, each calculus introduced in the literature so far exhibits some of the following features: non-standard readings of sequents[1] and non-standard introduction rules for logical operators (where the Łukasiewicz implication is a case in point).

The distinctive axiom of Łukasiewicz logic, namely $(A \rightarrow B) \rightarrow B \vdash A \vee B$,[2] is not *analytic-inductive* [10] (not even canonical) and it represents the main obstacle to a *uniform* and *modular* proof-theoretic treatment. Pivoting on an algebraic analysis of Łukasiewicz logic, we introduce a refinement of the general theory of display sequent calculi and algorithmic rule generation (as developed, for instance, in [6] and [10], respectively) aiming at surpassing this problem.

In particular, we rely on the fact that Łukasiewicz operators (see table below where the full language is considered) are not only normal operators, but also regular operators in the following sense (in [9] and [7] such operators are called 'double quasioperators'):

| | |
|---|---|
| normal binary diamond | normal binary box |
| $A \odot \mathbf{0} = \mathbf{0} = \mathbf{0} \odot A$ | $A \oplus \mathbf{1} = \mathbf{1} = \mathbf{1} \oplus A$ |
| $(A \vee B) \odot C = (A \odot C) \vee (B \odot C)$ | $(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C)$ |
| $C \odot (A \vee B) = (C \odot A) \vee (C \odot A)$ | $C \oplus (A \wedge B) = (C \oplus A) \wedge (C \oplus B)$ |
| $A \ominus \mathbf{1} = \mathbf{0} = \mathbf{0} \ominus A$ | $A \rightarrow \mathbf{1} = \mathbf{1} = \mathbf{0} \rightarrow A$ |
| $(A \vee B) \ominus C = (A \ominus C) \vee (B \ominus C)$ | $(A \vee B) \rightarrow C = (A \rightarrow C) \wedge (B \rightarrow C)$ |
| $C \ominus (A \wedge B) = (C \ominus A) \vee (C \ominus B)$ | $C \rightarrow (A \wedge B) = (C \rightarrow A) \wedge (C \rightarrow B)$ |
| regular binary diamond | regular binary box |
| $(A \vee B) \oplus C = (A \oplus C) \vee (B \oplus C)$ | $(A \wedge B) \odot C = (A \odot C) \wedge (B \odot C)$ |
| $C \oplus (A \vee B) = (C \oplus A) \vee (C \oplus B)$ | $C \odot (A \wedge B) = (C \odot A) \wedge (C \odot B)$ |
| $(A \wedge B) \rightarrow C = (A \rightarrow C) \vee (B \rightarrow C)$ | $(A \wedge B) \ominus C = (A \ominus C) \wedge (B \ominus C)$ |
| $C \rightarrow (A \vee B) = (C \rightarrow A) \vee (C \rightarrow B)$ | $C \ominus (A \vee B) = (C \ominus A) \wedge (C \ominus B)$ |

Exploiting the previous observation, we introduce a language expansion where the different "personalities" (normal versus regular) of the operators are fully-fledged and, in turn, it becomes possible

[1] E.g. the structural comma occurring in the antecedent and in the consequent of a sequent is interpreted as $\oplus$ in both cases, and the empty antecedent and the empty consequent of sequents is interpreted as $\mathbf{1}$ in both cases.

[2] Or any equivalent axiom in any complete fragment of Łukasiewicz logic.

introducing a sequent calculus with the so-called *relativized display property* (namely, every structure occurring in a derivable sequent is displayable). Moreover, all the logical introduction rules are standard and reflect the minimal order-theoretic properties of the operators, while the specific features of the logic are captured by so-called structural rules, so maintaining a neat division of labour that guarantees a modular treatment. At last, all the structural rules are automatically generated via (a specialisation of) the algorithm ALBA (to regular operators). Showing that the calculus enjoys (canonical) cut elimination is future work.

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# A representation theorem for a system of point-free geometry*

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In Gerla and Gruszczyński (2017) we put forward a system of geometry based on the primitive notions of *region*, *parthood* and *oval*, the last one being a counterpart of the well-known notion of *convex set*. Regions are interpreted as elements of a complete atomless Boolean algebra, parthood as the standard Boolean order, and ovals as elements of a distinguished set of regions.

The notion of *oval*[1] is very expressive from geometrical point of view, and by means of it we can define well-known standard notions. For example:

1. a *half-plane* is any element of the set $\mathbf{H} \subseteq \mathbf{O}^+$ closed for the Boolean complement,[2]
2. a line is a pair $\langle x, y\rangle$ of non-zero ovals that is maximal with respect to the pointwise order in $\mathbf{O}^+ \times \mathbf{O}^+$ (inherited from the standard pointwise order on the product of the algebra), $x$ and $y$ are *sides* of the line,
3. lines $L_1$ and $L_2$ are parallel iff they have disjoint sides,
4. a line $L$ *crosses* a region $x$ iff both sides of $L$ overlap $x$ (i.e., have the non-zero meet with $x$),
5. regions $x_1, \dots, x_n$ are *aligned* iff there is a line $L$ that crosses them all,
6. an *angle* is any region that is the meet of the sides of non-parallel lines, and a *stripe* is the non-zero meet of two half-planes that are sides of two parallel lines,
7. the *hull* of a region $x$ (in symbols: $\mathrm{hull}(x)$) is the infimum of all ovals that contain $x$.

With the concepts defined above in (Gerla and Gruszczyński, 2017) we formulated the following axioms for structures $\langle \mathbf{R}, \leq, \mathbf{O}\rangle$:

$$\langle \mathbf{R}, \leq\rangle \text{ is a complete atomless Boolean lattice.} \tag{O0}$$

$$\mathbf{O} \text{ is an algebraic closure system in } \langle \mathbf{R}, \leq\rangle \text{ containing } \mathsf{0}. \tag{O1}$$

$$\mathbf{O}^+ \text{ is dense in } \langle \mathbf{R}^+, \leq\rangle\,. \tag{O2}$$

$$\text{The sides of a line form a partition of } \mathsf{1}\,. \tag{O3}$$

$$\text{For any } a, b, c \in \mathbf{O}^+ \text{ which are not aligned there is a line which separates } a \text{ from } \mathrm{hull}(b + c). \tag{O4}$$

$$\text{If distinct lines } L_1 \text{ and } L_2 \text{ both cross an oval } a, \text{ then they split } a \text{ into at least three parts.} \tag{O5}$$

$$\text{No half-plane is part of any stripe or angle.} \tag{O6}$$

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[1] Let $\mathbf{O}$ be the set of all ovals of a given structure.

[2] For any set of regions $A$, $A^+ := A \setminus \{\mathsf{0}\}$, where $\mathsf{0}$ is the minimal element of the algebra.

The structure composed of regular open subsets of the two-dimensional Cartesian plane with ovals interpreted as its convex regions is a model of (O0)–(O6), therefore the system is consistent.

In (Gerla and Gruszczyński, 2017) we proved that in this system all axioms of a point-free system geometry by Śniatycki (1968) are provable, and thus using the results of the latter paper, by means the notions of *regions*, *part of* and *oval* we can define the standard geometrical concepts of *point* and *betweenness*, and moreover we can prove all the axioms of the betweenness fragment of geometry (i.e., we can capture affine geometry).

The intention of this talk is to present a sketch of the proof of the following representation theorem, which is an extension of the results from (Gerla and Gruszczyński, 2017):

**Theorem 1.** *For any point-free system of geometry satisfying axioms* (O0)–(O6) *there is a topological space* $\langle \Pi, \mathscr{O} \rangle$ *with a notion of convex set* $\mathbf{C}$*, such that* $\langle \mathbf{R}, \leq \rangle$ *is isomorphic with the algebra of regular open subsets of* $\Pi$ *via a certain function* $f$*, and for any region* $x$*:* $x \in \mathbf{O}$ *iff* $f(x) \in \mathbf{C}$.

Intuitively, points of the topological space are constructed from geometrical entities as equivalence classes of non-parallel half-planes. This goes along the intuition that a point on a plane can be identified with a pair of intersecting lines. Further, any pair of non-parallel half-planes determines a four-element partition of the unity of the Boolean algebra. Using this we can define an *internal* point $\alpha$ of a region $x$, by requiring that for every representative $P$ of this point (i.e., any pair of non-parallel half-planes that represents $\alpha$), all four elements of the partition determined by $P$ meet some oval $a$ that is a part of $x$ (in the sense that they have non-zero boolean meets with $a$). The idea is that we can take the family of all sets of internal points of ovals as a basis for the topological space from the theorem, and we can take the mapping taking regions to their internal points to establish the representation. In particular, we may prove that for any oval $a$ the set of its internal points is a convex set in the topological space, where a convex set is characterized by means of the betweenness relation in the standard way: as the set that together with any pair $\alpha$ and $\beta$ of its points contains all points lying between $\alpha$ and $\beta$.

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# Dualities from categorical first principles

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Many dualities of interest relate a category of algebras with some other, non-algebraic category: often a category of spaces. Classical Stone duality is a quintessential example of this phenomenon, describing the dual equivalence of the category of Boolean algebras with the category of Stone spaces. There are a multitude of Stone-type dualities which vary on this theme. A notable example is *Priestley duality*, the dual equivalence of the category of distributive lattices with that of Priestley spaces.

In this talk, we present our work developing a categorical framework for such dualities. Our goal is a categorically elegant approach, unifying and simplifying their construction. We motivate and describe our framework by showing how classical Stone duality and Priestley duality may be derived by way of the *ultrafilter* and *prime filter monads* respectively. Monads play a central role in the category-theoretic formulation of general (universal) algebra: in this context, they may be regarded as a generalisation of algebraic closure operators, allowing infinitary operations and arbitrary underlying objects replacing sets. In the late 1960s, Manes showed that the category of algebras for the ultrafilter monad is equivalent to the category of compact Hausdorff spaces [1]. In 1997, Flagg proved an analogous result: that the ordered counterpart of compact Hausdorff spaces, i.e. *compact pospaces*, are algebras for the prime filter monad — a monad on the category of partially ordered sets [2].

These monads are induced by adjoint functors of a particular type, and there are canonical comparison functors (contravariant) between the categories of Boolean algebras and compact Hausdorff spaces (in the case of the ultrafilter monad) and distributive lattices and compact pospaces (in the case of the prime filter monad). We rely on a key fact: the comparison functors for the ultrafilter and prime filter monads are contravariant fully faithful functors, and restricting to the essential image of such functors will always yield a duality. Here, we recover Stone and Priestley dualities respectively.

We will proceed to discuss current work refining our framework. This involves characterising conditions under which the comparison functor is full and faithful, with an eye to reconciling with Birkhoff's subdirect representation theorem.

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# On the double category of coalgebras

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Classically, two states of coalgebras of the same functor are behaviourally equivalent whenever they are identifiable by morphisms of coalgebras that share the same codomain. However, for quantitative systems it is often more reasonable to consider states with "close behaviour" instead, which leads to the notion of *behavioural distance* [1] and of metric bisimulation [4]. We show that the latter notion is captured by the first one; that is, the notion of similarity provided by a lax extension corresponding to a class of monotone predicate liftings coincides with the notion of behavioural distance provided by the lifting associated with the same class of predicate liftings. This is the missing link mentioned in [3] that makes is possible to incorporate the approach to similarity via lax extensions in the categorical frameworks described in [2] and [3]. In fact, we describe this connection at a high level of generality and argue that a natural double category of coalgebras for a lax double functor provides a suitable context to reason coalgebraically about various notions of *indistinguishability*. From this point of view we also recover the results from [5] and obtain a new result for (quasi) uniform spaces which complements the expressivity result for uniform spaces obtained in [3].

In this talk we report on joint work with Sergey Goncharov, Pedro Nora, Lutz Schröder and Paul Wild (Friedrich-Alexander-Universität Erlangen-Nürnberg).

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# Uniform Lyndon Interpolation for Basic Non-normal Modal and Conditional Logics

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In this talk, via a proof-theoretic method, we show that the non-normal modal logics $\mathsf{E}$, $\mathsf{M}$, $\mathsf{EN}$, $\mathsf{MN}$, $\mathsf{MC}$, $\mathsf{K}$, and their conditional versions, $\mathsf{CE}$, $\mathsf{CM}$, $\mathsf{CEN}$, $\mathsf{CMN}$, $\mathsf{CMC}$, $\mathsf{CK}$, in addition to $\mathsf{CKID}$ enjoy the uniform Lyndon interpolation property. This result in particular implies that these logics have uniform interpolation. Although for some of them the latter is known, the fact that they have uniform Lyndon interpolation is new. Also, the proof-theoretic proofs of these facts are new, as well as the constructive way to explicitly compute the interpolants that they provide. On the negative side, we show that the logics $\mathsf{CKCEM}$ and $\mathsf{CKCEMID}$ enjoy uniform interpolation but not uniform Lyndon interpolation. Moreover, we prove that the non-normal modal logics $\mathsf{EC}$ and $\mathsf{ECN}$ and their conditional versions, $\mathsf{CEC}$ and $\mathsf{CECN}$, do not have Craig interpolation, and whence no uniform (Lyndon) interpolation. This talk is based on a joint work with Amir Akbar Tabatabai and Rosalie Iemhoff. The non-normal modal part was published in WoLLIC 2021 [1] and the extended version with the conditional logics is submitted to its special issue of Journal of Logic and Computation.

In the rest of this abstract, we will discuss the details of the results. Set $\mathcal{L}_\Box = \{\wedge, \vee, \to, \bot, \Box\}$ as the language of modal logics and $\mathcal{L}_\triangleright = \{\wedge, \vee, \to, \bot, \triangleright\}$ as the language of conditional logics. The sets of positive and negative variables of a formula $\varphi \in \mathcal{L}$, denoted respectively by $V^+(\varphi)$ and $V^-(\varphi)$ are defined recursively as expected. Note that $V^+(\varphi \triangleright \psi) = V^-(\varphi) \cup V^+(\psi)$ and $V^-(\varphi \triangleright \psi) = V^+(\varphi) \cup V^-(\psi)$, for $\mathcal{L} = \mathcal{L}_\triangleright$. Define $V(\varphi) = V^+(\varphi) \cup V^-(\varphi)$. Lyndon interpolation (LIP) and Craig interpolation property (CIP) for logics are defined as usual. In the following, we define Uniform Lyndon interpolation (ULIP) and uniform interpolation (UIP) for logics.

**Definition 1.** A logic $L$ has *ULIP* if for any formula $\varphi \in \mathcal{L}$, atom $p$, and $\circ \in \{+, -\}$, there are $p^\circ$-free formulas, $\forall^\circ p\varphi$ and $\exists^\circ p\varphi$, such that $V^\dagger(\exists^\circ p\varphi) \subseteq V^\dagger(\varphi)$, $V^\dagger(\forall^\circ p\varphi) \subseteq V^\dagger(\varphi)$, for any $\dagger \in \{+, -\}$, and $L \vdash \forall^\circ p\varphi \to \varphi$ and $L \vdash \varphi \to \exists^\circ p\varphi$. Moreover, for any $p^\circ$-free formula $\psi$ if $L \vdash \psi \to \varphi$, then $L \vdash \psi \to \forall^\circ p\varphi$, and $L \vdash \exists^\circ p\varphi \to \psi$. A logic has *UIP* if it has all the mentioned properties, omitting $\circ, \dagger \in \{+, -\}$, everywhere.

The logic $\mathsf{E}$ is defined as the smallest set of formulas in $\mathcal{L}_\Box$ containing classical tautologies and closed under modes ponens and the rule $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}\,(E)$. Other non-normal logics can be defined by adding the following modal axioms to $\mathsf{E}$:

$$\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi \quad (\mathrm{M}), \qquad \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi) \quad (\mathrm{C}), \qquad \Box\top \quad (\mathrm{N}).$$

*Speaker.

We consider the following non-normal modal logics: $\mathsf{EN} = \mathsf{E} + (N), \mathsf{M} = \mathsf{E} + (M), \mathsf{MN} = \mathsf{M} + (N), \mathsf{MC} = \mathsf{M} + (C), \mathsf{K} = \mathsf{MC} + (N), \mathsf{EC} = \mathsf{E} + (C)$, and $\mathsf{ECN} = \mathsf{EC} + (N)$. Similarly, for conditional logics, $\mathsf{CE}$ is defined as the smallest set of formulas in $\mathcal{L}_{\triangleright}$ containing classical tautologies and closed under modes ponens and $\dfrac{\varphi_0 \leftrightarrow \varphi_1 \qquad \psi_0 \leftrightarrow \psi_1}{\varphi_0 \triangleright \psi_0 \to \varphi_1 \triangleright \psi_1}\,(CE)$. The other conditional logics are defined by adding the following conditional axioms to $\mathsf{CE}$:

$$(\varphi \triangleright \psi \wedge \theta) \to (\varphi \triangleright \psi) \wedge (\varphi \triangleright \theta) \quad \text{(CM)}, \quad (\varphi \triangleright \psi) \wedge (\varphi \triangleright \theta) \to (\varphi \triangleright \psi \wedge \theta) \quad \text{(CC)},$$

$$\varphi \triangleright \top \quad \text{(CN)}, \quad (\varphi \triangleright \psi) \vee (\varphi \triangleright \neg\psi) \quad \text{(CEM)}, \quad \varphi \triangleright \varphi \quad \text{(ID)}.$$

We consider the following conditional logics: $\mathsf{CEN} = \mathsf{CE} + (CN), \mathsf{CM} = \mathsf{CE} + (CM), \mathsf{CMN} = \mathsf{CM} + (CN), \mathsf{CMC} = \mathsf{CM} + (CC), \mathsf{CK} = \mathsf{CMC} + (CN), \mathsf{CEC} = \mathsf{CE} + (CC), \mathsf{CECN} = \mathsf{CEC} + (CN), \mathsf{CKID} = \mathsf{CK} + (ID), \mathsf{CKCEM} = \mathsf{CK} + (CEM)$, and $\mathsf{CKCEMID} = \mathsf{CKCEM} + (ID)$.

**Theorem 2.** $(ULIP)$ *The logics* $\mathsf{E}$*,* $\mathsf{M}$*,* $\mathsf{MC}$*,* $\mathsf{EN}$*,* $\mathsf{MN}$*,* $\mathsf{K}$*, their conditional versions* $\mathsf{CE}$*,* $\mathsf{CM}$*,* $\mathsf{CMC}$*,* $\mathsf{CEN}$*,* $\mathsf{CMN}$*,* $\mathsf{CK}$*, and the conditional logic* $\mathsf{CKID}$ *have ULIP and hence UIP and LIP.*

$(UIP)$ *The logics* $\mathsf{CKCEM}$ *and* $\mathsf{CKCEMID}$ *enjoy UIP and hence CIP.*

$(Negative)$ *The logics* $\mathsf{EC}$ *and* $\mathsf{ECN}$ *and their conditional versions* $\mathsf{CEC}$ *and* $\mathsf{CECN}$ *do not have CIP. As a consequence, they do not have UIP or ULIP. Moreover, the logics* $\mathsf{CKCEM}$ *and* $\mathsf{CKCEMID}$ *do not enjoy ULIP.*

**Proof sketch.** To show our result, we use the sequent calculi for these logics. For modal logics the sequent calculi are defined in [2] and the cut elimination theorem is proved. For conditional logics, we introduce the sequent calculi and prove that the cut rule can be eliminated (the sequent calculi for the logics $\mathsf{CK}$, $\mathsf{CKID}$, $\mathsf{CKCEM}$, and $\mathsf{CKCEMID}$ were studied in [3]). To prove ULIP for these logics we extend the notion to sequent calculi. It is easy to see that ULIP for a sequent calculus implies that the corresponding logic has ULIP. Then, using the natural notion of weight on formulas and sequents we can define a well-ordering on the sequents. We use this well-ordering to define the uniform interpolants and prove the desired properties by induction on this well-ordering.

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# Quantum logics as algebras for monads

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In [7], Kalmbach proved the following theorem.

**Theorem 1.** *Every bounded lattice $L$ can be embedded into an orthomodular lattice $K(L)$.*

The proof of the theorem is constructive, $K(L)$ is known under the name *Kalmbach extension* or *Kalmbach embedding*. In [8], Mayet and Navara proved that Theorem 1 can be generalized: every bounded poset $P$ can be embedded into an orthomodular poset $K(P)$. In 2004 Harding explained where does the Kalmbach construction come from:

**Theorem 2.** *[3, Theorem 16] $K$ is a functor left adjoint to the forgetful functor from the category of orthomodular posets to the category of bounded posets.*

However, as remarked by Harding in the same paper, $K$ does not restrict to a functor between the category of orthomodular lattices and the category of bounded lattices. As every adjunction, the adjunction from Theorem 2 induces a monad on the category of orthomodular posets, which we call *the Kalmbach monad*.

In their seminal paper [2], Foulis and Bennett introduced the notion of an *effect algebra.*

**Theorem 3.** *[6] The category of effect algebras is equivalent to the category of algebras for the Kalmbach monad.*

In other words, the forgetful functor from the category of effect algebras to the category of bounded posets in monadic. This theorem means the category of effect algebras in inherently present in the forgetful functor from orthomodular posets to bounded posets.

In [1], Dvurečenskij and Vetterlein introduced a non-commutative generalization of effect algebras, called *pseudo-effect algebras.*

**Theorem 4.** *[4] The forgetful functor from the category of pseudo effect algebras to the category of bounded posets is monadic.*

An important subcategory of the effect algebras is the category of $\omega$-*effect algebras*, in which sums of certain countable families of elements are required to exist.

**Theorem 5.** *[9] The forgetful functor from the category of $\omega$-effect algebras to the category of bounded posets is monadic.*

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Unlike the other adjunctions here, the adjunction between orthomodular posets and bounded posets from Theorem 2 is not monadic. This leads to a natural question: is the category of orthomodular posets isomorphic to a category of algebras for a monad on some category, in a nontrivial way? The following theorem answers this questions in the positive.

**Theorem 6.** *[5] The forgetful functor from the category of orthomodular posets to the category of bounded posets with involution is monadic.*

The proofs of the right-adjointness of the forgetful functor in Theorems 4, 5 and 6 use the general adjoint functor theorem, hence these proofs are non-constructive.

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# On varieties of residuated po-magmas and the structure of finite ipo-semilattices

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In [6] *po-algebras* are defined as partially ordered sets with operations that are either order-preserving or order-reversing in each argument, and a variety of po-algebras is a class of similar po-algebras defined by equations or inequations.

A *residuated partially ordered magma* or *rpo-magma* $\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a partially-ordered set $(A, \leq)$ with a binary operation $\cdot$ and two *residuals* that satisfy for all $x, y, z \in A$

$$\text{(res)} \quad xy \leq z \iff x \leq z/y \iff y \leq x\backslash z$$

The operation $x \cdot y$ is usually written $xy$. Residuation ensures that $x\backslash y$ and $y/x$ are order-reversing in the denominator ($x$ position) and order-preserving in the numerator, while $xy$ is order-preserving in both arguments. Since (res) is equivalent to the inequations $x \leq xy/y$, $(z/y)y \leq z$, $y \leq x\backslash xy$, $x(x\backslash z) \leq z$ it follows that rpo-magmas are a variety of po-algebras. Although rpo-magmas are very general, (res) imposes restrictions on the posets that can occur.

**Theorem 1.** *In an rpo-magma every connected component of $\leq$ is up-directed and down-directed, hence for finite rpo-magmas every connected component is bounded.*

The equivalence relation on a poset that has each connected component as an equivalence class is a congruence on a rpo-magma, and the quotient algebra is a quasigroup with the discrete order (i.e. $\leq$ is the equality relation). Conversely, from any quasigroup $Q$ and a pair-wise disjoint family of *bounded* posets $A_q$ for $q \in Q$, one can construct an rpo-magma with poset $\bigcup_{q\in Q} A_q$.

A *rpo-semigroup* or *Lambek algebra* is a rpo-magma where $\cdot$ is associative. A *unital* rpo-magma has a constant 1 such that $x1 = x = 1x$, and a *rpo-monoid* is a unital rpo-semigroup. A *residuated lattice-ordered magma* $(A, \wedge, \vee, \cdot, \backslash, /)$ (or *r$\ell$-magma* for short) is a rpo-magma for which the partial order is a lattice order. A $r\ell$-monoid is more commonly called a *residuated lattice*.

An *involutive partially-ordered magma* or *ipo-magma* is of the form $(A, \leq, \cdot, \sim, -)$ such that $(A, \leq)$ is a poset, $\cdot$ is a binary operation, the *left and right linear negations* $\sim, -$ are an *involutive pair*, i.e., ${\sim}{-}x = x = -{\sim}x$, $x \leq {\sim}y \iff y \leq -x$, and for all $x, y, z \in A$

$$\text{(ires)} \quad xy \leq z \iff x \leq -(y \cdot {\sim}z) \iff y \leq {\sim}(-z \cdot x).$$

It follows that $\sim, -$ are both order-reversing. The axiom (ires) shows that every ipo-magma is term-equivalent to a rpo-magma, but it is often convenient to use the equivalent formulation

$$\text{(rotate)} \quad xy \leq z \iff y \cdot {\sim}z \leq {\sim}x \iff -z \cdot x \leq -y.$$

The variety of ipo-monoids includes all partially ordered groups [1], where ${\sim}x = -x = x^{-1}$.

**Lemma 2.** *Let $\mathbf{A} = (A, \leq, \cdot, \sim, -)$ be a poset with a binary and two unary operations. (1) If $\cdot$ is idempotent (i.e. $xx = x$) and $\mathbf{A}$ satisfies* (rotate) *then $\mathbf{A}$ is an ipo-magma. (2) If an ipo-magma is idempotent or unital, and $\cdot$ is commutative then ${\sim}x = -x$.*

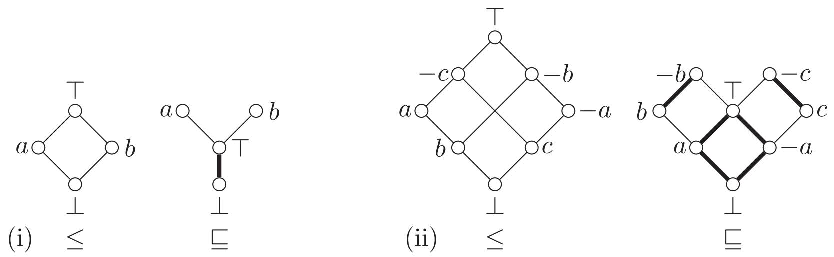


Figure 1: (i) The smallest i$\ell$-semilattice that is not unital, and (ii) the smallest ipo-semilattice that is not lattice-ordered. Note that $-\top = \bot$.

In a commutative idempotent ipo-semigroup there is another semilattice order, called the *multiplicative order*, that is defined by $x \sqsubseteq y \iff xy = x$, and these po-algebras are called *ipo-semilattices*. The most prominent examples of ipo-semilattices are Boolean algebras $(A, \cdot, +, -)$, where $x + y = -(-x \cdot -y)$. They arise as the case when the partial order $\leq$ and the semilattice order $\sqsubseteq$ coincide. More generally, ipo-semilattices are determined by the two relations $\leq$, $\sqsubseteq$ and the involution $-$ (see e.g. Figure 1).

Note that an element $t$ in an ipo-semilattice is the multiplicative identity if and only if $t$ is the top element in the multiplicative order, hence an ipo-semilattice is unital if and only if the multiplicative order has a top element.

We give a description of the structure of finite i$\ell$-semilattices, and provide some partial structural results for finite ipo-semilattices. For finite commutative idempotent involutive residuated lattices (i.e. unital i$\ell$-semilattices) a structural description has been given in [4]. We present a more global approach for ipo-semilattices based on Płonka sums of Boolean algebras. Similar methods have been used in [3] to describe the structure of odd and of even involutive $\mathrm{FL}_e$-chains. Inspired by the duality for involutive bisemilattices [2], we also give a more compact dual description of finite ipo-semilattices based on semilattice direct systems of partial maps between sets. We present an algorithm to construct finite ipo-semilattices from their dual description, and an algorithm to enumerate the dual objects up to isomorphism. Some of our investigations were aided by computations using Prover9 and Mace4 [5].

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# Filtral pretoposes and compact Hausdorff locales

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In their recent work [3] Marra and Reggio charactrize the category of compact Hausdorff spaces as the unique, up to equivalence, non-trivial, well-pointed, filtral pretopos with set-indexed copowers of its terminal object. Recall that a pretopos is a category which has disjoint and universal sums, a feature typical of categories of spaces, while it has pullback-stable image factorizations and every equivalence relation has a coequalizer and is the kernel pair of it, a feature typical of algebraic categories.

In view of the significant role that compact Hausdorff locales have in the development of mathematics internally in a topos it would be interesting to know if the category of compact Hausdorff locales admits a similar "pointless" characterization. The approach adopted in [3] has the deficit, from our perspective, that it stresses the role of points right from the beginning. It has though the advantage of introducing the key concept of filtrality, which is fundamental for our approach too. Filtrality is the property of having enough objects whose lattice of subobjects is the dual of a Stone frame. Without resorting to the classically valid equivalence with the respective topological spaces, one can show that the category CHLoc of compact Hausdorff locales is a pretopos [2] and moreover it is filtral ([1], D4.6.8).

We show that, for any filtral pretopos, there is a functor to CHLoc, namely the one that assigns to an object of such a pretopos the lattice of subobjects of it with its dual order. For that we needed to show first that a closed quotient of a compact Hausdorff locale is Hausdorff, a result that may have its own independent interest. Indeed one has

**Theorem 1.** *If $f\colon Y \to X$ is a closed surjection of locales and $Y$ is compact Hausdorff then $X$ is compact Hausdorff (and hence the surjection is proper).*

**Proof:** Every compact Hausdorff locale admits a proper surjection from a Stone locale and the composite $fe$ of $f$ with a proper surjection $e$ is proper iff $f$ is proper, so assume that $Y$ is Stone. As such it is subfit, i.e every open sublocale of it is intersection of closed ones. Being compact Hausdorff it is also normal. Hence $X$ is also compact and normal. It suffices, that $X$ is also subfit, because then following [4] Proposition 4.4, $X$ is regular, hence $X$ is compact Hausdorff. The result follows from the next proposition. □

**Proposition 2.** *If $f\colon Y \to X$ is a closed surjection of locales and $Y$ is subfit then $X$ is subfit.*

**Proof:** If the nucleus $j = u \to -$ corresponds to an open sublocale $U$ of $X$ then $f^{-}j = f^{*}u \to -$ corresponds to the inverse image of $U$ in $Y$. Since $Y$ is subfit we have that $f^{*}u \to - = \bigwedge_i (v_i \vee -)$ in the frame of nuclei on $OY$. Hence its direct image is $f_{+}f^{-}j = f_{+}(\bigwedge_i v_i \vee -) =$

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$\bigwedge_i f_+(v_i \vee -)$. Each $f_+(v_i \vee -)$ is closed by the assumption of closedness of $f$. On the other hand, while for each $j \in NX$, in general $j \leq f_+f^-j$, for $j = u \to -$ we moreover have $f_+f^-j = f_*(f^*u \to f^*-) \leq j = u \to -$ because if $w \leq f_*(f^*u \to f^*v)$, then $f^*w \leq f^*u \to f^*v$, equivalently $f^*(w \wedge u) \leq f^*v$, so we conclude that $w \leq u \to v$ by the surjectivity of $f$. $\square$

Filtrality of a pretopos $\mathcal{K}$ gives that, for each $X \in \mathcal{K}$ the lattice of its subobjects, with its dual order, is the frame of a closed (because of Frobenious reciprocity) quotient of a Stone locale, so the assignment $X \mapsto \mathrm{Sub}(X)^{op}$ is the object part of a functor from the filtral pretopos $\mathcal{K}$ to the category of compact Hausdorff locales CHLoc, with image $f[-]$ of subobjects as direct image of the locale map.

**Theorem 3.** *For a filtral pretopos $\mathcal{K}$, the functor $\mathrm{Sub}(-)^{op}\colon \mathcal{K} \to \mathrm{CHLoc}$ is full on subobjects, faithful, preserves (regular) epis and equalizers. Assuming further that the product $S = S_1 \times S_2$ of two filtral objects is filtral, the map $B_1 \coprod B_2 \to B$ involving the respective boolean algebras of complemented subobjects is injective and the unique map from $\mathrm{Sub}\,\mathbf{1}$ to the terminal locale (which is compact Hausdorff) is a surjection, then it preserves finite products as well.*

**Proof:** Concerning preservation of equalizers, upon which faithfulness also hinges, for a pair of maps $f, g : Y \to Z$ in $\mathcal{K}$ with equalizer $X \to Y$, by the description of equalizers of Hausdorff locales in [5], the equalizer of $f[-]$, $g[-]$ is given as $\downarrow(\bigwedge\{f^{-1}[S] \vee g^{-1}[\sim S] \in OX \mid S \leq Z\})$ (taking into account the existence of dual pseudo-complements in $\mathrm{Sub}(Z)$.) In the less obvious direction, $\mathrm{Sub}(X)^{op}$ is contained in the equalizer because $X$ is below each $f^{-1}[S] \wedge g^{-1}[\sim S]$. Indeed, for each $S \leq Z$ we have $X \wedge f^{-1}[S] = X \wedge g^{-1}[S]$ (because $T \leq X \wedge f^{-1}[S]$ iff $T \leq X$ and $T \leq f^{-1}[S]$, equivalently $T \leq X$ and $f[T] \leq S$, iff $T \leq X$ and $g[T] \leq S$, or, $T \leq X$ and $T \leq g^{-1}[S]$.) Then

$$\begin{aligned} X \wedge (f^{-1}[S] \vee g^{-1}[\sim S]) &= (X \wedge f^{-1}[S]) \vee (X \wedge g^{-1}[\sim S]) \\ &= (X \wedge f^{-1}[S]) \vee (X \wedge f^{-1}[\sim S]) \\ &= X \wedge f^{-1}[S \vee \sim S]) = X \wedge Y = X \qquad \square \end{aligned}$$

If $\mathcal{K}$ has copowers of 1, and in the base topos Stone locales have enough points, one gets that the functor is also covering, thus an equivalence. The characterization of [3] is recovered this way.

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# Frobenius structures in autonomous categories

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In a series of successive works the following result is established:

**Theorem 1** (See [7, 4, 5, 10, 9]). *The quantale $[L, L]$ of sup-preserving endomaps of a complete lattice $L$ is a Frobenius quantale if and only if $L$ is a completely distributive lattice.*

We give a proof of this result that relies on the $*$-autonomous structure of $\mathsf{SLatt}$, the category of complete lattices and sup-preserving maps. In doing so, we generalise this result to arbitrary $*$-autonomous categories. Recall that an object $A$ of an autonomous category $\mathcal{V} = (V, I, \otimes, \alpha, \lambda, \rho, [-, -], ev)$ is *nuclear* if the canonical map $\mathtt{mix} : A^* \otimes A \to [A, A]$ is an isomorphism, where $A^*$, the dual of $A$, is the internal hom $[A, I]$. We rely on the following characterization of completely distributive lattices:

**Theorem 2** (See [8, 6]). *Nuclear objects in* $\mathsf{SLatt}$ *are exactly the completely distributive lattices.*

A main tool that we use is the notion of dual pair:

**Definition 3.** A dual pair in a monoidal category $\mathcal{V}$ is a triple $(A, B, \epsilon)$, with $A, B$ objects of $\mathcal{V}$ and $\epsilon : A \otimes B \longrightarrow I$, yielding via Yoneda natural isomorphisms

$$\hom(X, B) \simeq \hom(A \otimes X, I) \text{ and } \hom(X, A) \simeq \hom(X \otimes B, I)\,.$$

We informally say that $(A, B)$ is a dual pair, leaving aside the arrow $\epsilon$. Clearly, $(A, A^*)$ is a dual pair in any $*$-autonomous category. This notion provides the framework by which to study objects that are dual to each other only up to isomorphism: for example $(A^* \otimes A, [A, A])$ is a dual pair in any $*$-autonomous category and, for any complete lattice $L$, $(L, L^{op})$ is a dual pair in $\mathsf{SLatt}$. Some elementary properties of dual pairs are immediate, for instance, if $(A, B)$ is a dual pair, then $A$ and $B$ are both reflexive, that is, isomorphic to their double dual.
If $(A, B)$ is a dual pair and $A$ is a semigroup, then $A$ acts on $B$ on the left and on the right. The left and right actions, written here $\alpha^\ell$ and $\alpha^\rho$, correspond, in the category $\mathsf{SLatt}$, to the two implications of a quantale, see e.g. [7, 3]. We define then generalized Frobenius quantales in arbitrary autonomous categories as follows:

**Definition 4.** A *Frobenius structure* is a tuple $(A, B, \mu_A, l, r)$ where $(A, B)$ is a dual pair, $(A, \mu_A)$ is a semigroup, $l$ and $r$ are adjoint invertible maps from $A$ to $B$ such that the diagram below on the left commutes:

$$\begin{tikzcd} A \otimes A \arrow[r, "A\otimes r"] \arrow[d, "l\otimes A"'] & A \otimes B \arrow[d, "\alpha^\ell_A"] \\ B \otimes A \arrow[r, "\alpha^\rho_A"'] & B \end{tikzcd} \qquad \begin{tikzcd} B \otimes B \arrow[r, "B\otimes l^{-1}"] \arrow[d, "r^{-1}\otimes B"'] & B \otimes A \arrow[d, "\alpha^\rho"] \\ A \otimes B \arrow[r, "\alpha^\ell"'] & B. \end{tikzcd} \tag{1}$$

By saying that $l$ and $r$ are adjoint, we mean that their transposes differ by a symmetry: $\epsilon \circ (A \otimes l) = \epsilon \circ (A \otimes r) \circ \sigma_{A,A}$. Definitions of Frobenius structures in various kind of monoidal

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categories already exist in the literature [1, 11, 3, 4]. In these works the units (and co-units) play an important role. Following [2], where we have argued that neither dualizing elements nor units are needed in order to define Frobenius quantales, Definition 4 does not require units. When $(A, \mu_A)$ is a semigroup in $\mathsf{SLatt}$, that is, a quantale, and $B = A^{op}$, the arrows $l$ and $r$ play the role of the two negations in a Frobenius quantales, they are adjoint in that they form a Galois connection. The diagram on the left of (1) may be understood as the equation $y\backslash^{\perp}x = y^{\perp}/x$ linking negations and implications.
Couples of quantales, as defined in [4], are apparently the most similar to the Frobenius structures defined here. For a couple of quantales, however, only one negation (not necessarly invertible) is considered and the right diagram of (1) is required to be commutative; once more, for the negation to be classical, one requires the existence of a dualizing element and thus of a unit. Definition 4 does not mention dualizing elements and implies the commutativity of the right diagram of (1). If we let $\mu_B$ be the diagonal of this diagram, then we have:

**Lemma 5.** *If $(A, B, \mu_A, l, r)$ is a Frobenius structure, then so is $(B, A, \mu_B, r^{-1}, l^{-1})$.*

It is now immediate to derive the following:

**Theorem 6.** *If $A$ is nuclear, then there is a map $l$ such that $([A, A], \circ, [A, A]^*, l, l)$ is a Frobenius structure.*

Indeed, $A^* \otimes A$ is canonically a semigroup and if the canonical map $\mathtt{mix} : A^* \otimes A \longrightarrow [A, A]$ is invertible, then $(A^* \otimes A, [A, A], \mu_{A^*\otimes A}, \mathtt{mix}, \mathtt{mix})$ is a Frobenius structure. We derive the theorem, since Frobenius structures are closed up to isomorphism and using Lemma 5. Theorem 6 is actually an instance of Corollary 3.3 in [11]. However, it can be further generalised: if $\mathtt{mix}$ is not invertible but the underlying $*$-autonomous category has some nice factorisation system, then the image of $\mathtt{mix}$ is the support of a Frobenius structure. This is a consequence of the following statement:

**Theorem 7.** *Let $\mathcal{V}$ be a $*$-autonomous category with a factorization system. Let $(A, \mu_A)$ be a semigroup and $(A, B)$ be a dual pair. Let $f : A \longrightarrow B$ be a map, put $\psi_A = \epsilon \circ (A \otimes f)$ and suppose that $\psi_A = \psi_A \circ \sigma_{A,A}$. Factor $f$ as $f = m \circ e$ with $e : A \longrightarrow C$ epi and $m : C \longrightarrow B$ mono. If $C$ is a magma with multiplication $\mu_C$ and $e$ is a magma homomorphism, then there exist maps $\psi_C : C \otimes C \longrightarrow I$ and $g : C \longrightarrow C^*$, transposing into each other, making $(C, C^*, \mu_C, g, g)$ into a Frobenius structure.*

The converse of Theorem 6 actually holds if we add another condition.

**Definition 8.** An objet $A$ of a monoidal category is *pseudo-affine* if the tensor unit $I$ embeds into $A$ as a retract. A monoidal category is *pseudo-affine* if every object which is not terminal nor initial is affine.

For example, the category $\mathsf{SLatt}$ is pseudo-affine.

**Theorem 9.** *In a $*$-autonomous category, if $A$ is a pseudo-affine object and the canonical monoid $([A, A], \circ)$ is part of a Frobenius structure, then $A$ is nuclear.*

The proof of this theorem relies on the following ideas. If $A, B$ are pseudo-affine, then the following statement holds:

**Lemma 10.** *If $A \otimes f = g \otimes B : A \otimes X \otimes B \longrightarrow A \otimes Y \otimes B$, then there exists $h : X \longrightarrow Y$ such that $f = h \otimes B$ and $g = A \otimes h$.*

This lemma is applied to the dual pair $([A, A], A^* \otimes A)$ as follows: since in this case $\alpha^\rho = A^* \otimes ev_{A,A}$, the diagonal of the diagram on the right of (1) is of the form $A^* \otimes f$. Considering the explicit form of $\alpha^\ell$, we also deduce that this same diagonal is of the form $g \otimes A$. Since both $A$ and $A^*$ are pseudo-affine, we deduce, by the last lemma, the existence of a map $\epsilon : A \otimes A^* \longrightarrow I$. Since $A^* \otimes A$ is isomorphic as a semigroup to $[A, A]$, it is also unital, thereby there exists a map $\eta : I \longrightarrow A^* \otimes A$. Since $A$ and $A^*$ are pseudo-affine, tensoring with them is faifthul and we deduce from the monoid diagrams for $A^* \otimes A$ that $(A, A^*, \eta, \epsilon)$ is an adjunction. We therefore deduce Theorem 9 from the fact that nuclear objects in an autonomous category are exactly the adjoints, left or right, since the category is symmetric.

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# Baker-Beynon duality beyond finite presentations

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Baker-Beynon duality is a fundamental result in the theory of abelian lattice-ordered groups ($\ell$-groups, for short) and vector lattices. In [1, 2], the category of finitely-presented vector lattices is proved to be equivalent to the one of *polyhedral cones* and piecewise (homogeneous) linear maps among them —the case of $\ell$-groups being slightly more complicated, as it involves polyhedral cones with *rational vertices* and maps with *integer* coefficients.

In [3] the authors propose a general framework in which many dualities, including Baker-Beynon duality, can be set. One starts with picking an arbitrary object in a category with some mild properties. The object induces a contravariant adjunction with another category to be thought of as category of *spaces*. Then a number of results, which are parametric on the arbitrary choice of the object, help characterise the fixed points of the adjunction, i.e., the ones for which the adjunction restricts to a duality. Baker-Beynon duality can be obtained from this framework setting the fixed object to be $\mathbb{R}$ and restricting to finitely generated objects. In this case the adjunction will fix only the Archimedean vector lattices (or $\ell$-groups), as they are exactly the subdirect products of $\mathbb{R}$. However, if one chooses a suitable ultrapower of $\mathbb{R}$, many more objects will be left fixed by the adjunction. The reason is to be found in the following result, which is an easy consequence of quantifier elimination for vector lattices and divisible ordered groups, respectively.

**Theorem 1.** *For every cardinal $\alpha$ there exists an ultrapower of $\mathbb{R}$ on an $\alpha$-regular ultrafilter, in which all linearly ordered vector lattices of cardinality smaller than $\alpha$ embed. The same is true for linearly ordered groups.*

We now provide more details on the general adjunctions induced by $\alpha$-regular ultrapowers of $\mathbb{R}$ and the restricted dualities. Hereafter, $\mathsf{V}$ can be taken to be either the variety of $\ell$-groups or Riesz spaces and $\mathcal{U}$ denotes invariably the $\alpha$-regular ultrapower of $\mathbb{R}$ —in the appropriate language— given by Theorem 1. We denote by $\mathscr{F}^{\ell}_{\kappa}$ the free $\kappa$-generated algebra in $\mathsf{V}$. Following the general framework of [3], for any $T \subseteq \mathscr{F}^{\ell}_{\kappa}$ and $S \subseteq \mathcal{U}^{\kappa}$, we define the following operators.

$$\mathbb{V}_{\mathcal{U}}(T) = \{x \in \mathcal{U}^{\kappa} \mid t(x) = 0 \text{ for all } t \in T\}, \qquad \mathbb{I}_{\mathcal{U}}(S) = \{t \in \mathscr{F}^{\ell}_{\kappa} \mid t(x) = 0 \text{ for all } x \in S\}.$$

The operators $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$ form a Galois connection. Upon defining the appropriate notion of arrows between spaces, $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$ can be lifted to a contravariant adjunction. A function $f\colon \mathcal{U}^n \to \mathcal{U}$ is called *definable* if there exists a term $t$ in the language on $\mathsf{V}$ such that $f(p) = t(p)$ for all $p \in \mathcal{U}^n$. The definition easily generalises to functions from $S \subseteq \mathcal{U}^{\mu}$ into $S' \subseteq \mathcal{U}^{\nu}$, with $\mu$ and $\nu$ cardinals. Let $\mathsf{G}_{\mathsf{def}}$ be the category of subsets of $\mathcal{U}^{\kappa}$, with $\kappa$ ranging among all cardinals, and definable maps among them.

The operators $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$ induce functors between $\mathsf{G}_{\mathsf{def}}$ and $\mathsf{V}$ as follows. For any subset $S \subseteq \mathcal{U}^{\kappa}$ and for any algebra in $\mathsf{V}$ (assumed to be presented in the form $\mathscr{F}^{\ell}_{\kappa}/J$),

$$\mathscr{I}(S) = \mathscr{F}^{\ell}_{\kappa}/\,\mathbb{I}_{\mathcal{U}}(S), \quad \mathscr{V}(\mathscr{F}^{\ell}_{\kappa}/J) = \mathbb{V}_{\mathcal{U}}(J)\,.$$

*Speaker.

We omit the definition of $\mathscr{I}$ and $\mathscr{V}$ on arrows, as it is more technical and not necessary in this context. By [3, Corollary 4.8] the functors $\mathscr{V}$ and $\mathscr{I}$ form a contravariant adjunction.

The fixed points of the adjunction easily correspond to the fixed points of the compositions of the operators $\mathbb{V}_{\mathcal{U}}$ and $\mathbb{I}_{\mathcal{U}}$. Regarding the algebraic side, [3, Theorem 4.15] guarantees that the ideals $J$ for which $\mathbb{I}_{\mathcal{U}} \circ \mathbb{V}_{\mathcal{U}}(J) = J$ holds are exactly the ones that can be obtained as intersections of ideals of the form $\mathbb{I}_{\mathcal{U}}(\{a\})$ for some $a \in A$. [3, Theorem 4.15] implies that an ideal of $\mathscr{F}^{\ell}_{\kappa}$ has the form $\mathbb{I}_{\mathcal{U}}(\{a\})$ if and only if the quotient over it embeds in $\mathcal{U}$. Since both vector lattices and $\ell$-groups are subdirect products of linearly ordered ones, an application of Theorem 1 proves that all the objects (of cardinality at most $\alpha$) in the algebraic side of the adjunction are left fixed.

As per the fixed points in $\mathsf{G}_{\mathsf{def}}$, it is readily seen that they are the closed subspaces of $\mathcal{U}^{\kappa}$ under a Zariski-like topology given by the following closed sets:

$$\mathbb{V}_{\mathcal{U}}(T) = \{x \in \mathcal{U}^{\kappa} \mid t(x) = 0 \text{ for all } t \in T\} \text{ for } T \text{ ranging among subsets of } \mathscr{F}^{\ell}_{\kappa}. \tag{1}$$

Summing up, if $\mathsf{V}_{\alpha}$ denotes the full subcategory of $\mathsf{V}$ whose objects have cardinality smaller than $\alpha$, we obtain the following duality theorem.

**Theorem 2.** *There is a dual equivalence between the category* $\mathsf{V}_{\alpha}$ *and the full subcategory* $\mathsf{K}$ *given by the closed objects in* $\mathsf{G}_{\mathsf{def}}$.

In addition to describing the dual categories to the classes of *all* Riesz spaces and $\ell$-groups, Theorem 2 also enables the use of tools of non-standard analysis in the study of these structures. We give an example below, others can be found in another submitted abstract by the same authors.

Let us assume that $\mathcal{U} = \mathbb{R}^I/\mathcal{F}$ and write $[(r_i)]$ for the equivalence classes in $\mathcal{U}$. Recall that, for any $A \subseteq \mathbb{R}$ the *enlargment* of $A$ is defined as follows:

$$[(r_i)] \in {}^*A \text{ if and only if } \{i \in I \mid r_i \in A\} \in \mathcal{F}.$$

General tools of nonstandard analysis show that basic closed set in the Zariski topology of (1), i.e. the set of the form $\mathbb{V}_{\mathcal{U}}(f)$ are enlargements of the analogous $\mathbb{V}_{\mathbb{R}}(f)$ in $\mathbb{R}^n$.

**Theorem 3.** *Let $k$ be a cardinal and let $J$ be an ideal of $\mathscr{F}^{\ell}_{\kappa}$.*

1. *$G \cong \mathscr{F}^{\ell}_{\kappa}/J$ is linearly ordered if and only if $\mathbb{V}_{\mathcal{U}}(J)$ is the closure of a point.*
2. *$G \cong \mathscr{F}^{\ell}_{\kappa}/J$ is semisimple if and only if $\mathbb{V}_{\mathcal{U}}(J)$ is the enlargement of a closed cone.*
3. *$G \cong \mathscr{F}^{\ell}_{\kappa}/J$ is finitely presented if and only if $\mathbb{V}(J)$ is the enlargement of a closed polyhedral cone – rational when $G$ is an $\ell$-group.*

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# Interrogative agendas, categorization, and decision-making

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The ongoing work we wish to discuss is based on the preliminary results in [1], where we initiate a line of research aimed at formally modelling various types of decision-making processes in terms of categorization processes.

In this work, we explore the role of the epistemic stances of different agents (decision-makers) played in the decision-making processes. We model those epistemic stances as *interrogative agendas* [2], a notion introduced in epistemology and formal philosophy indicating the set of questions individual agents (or groups of agents) are interested in, or what they consider relevant for deciding, relative to a certain circumstance (independently of whether they utter the questions explicitly). Interrogative agendas might differ for the same agent in different moments or in different contexts; for instance, my interrogative agenda when I have to decide which car to buy will be different from my interrogative agenda when I listen to a politician's speech. Deliberation and negotiation processes can be understood in terms of whether and how decision-makers/negotiators succeed in modifying their own interrogative agendas or those of their counterparts, and the outcomes of these processes can be described in terms of the "common ground" agenda thus reached.

An influential approach in logic [4] represents questions as equivalence relations over a suitable set of possible worlds $W$. When ordered by inclusion, the set of equivalence relations on any set $W$ is a complete (possibly non-distributive) lattice $E(W)$. Although the lattices $E(W)$ described above are in general not distributive, they resemble the powerset algebras in some important respects, for instance in their being completely join-generated and meet-generated by their atoms and co-atoms, respectively.

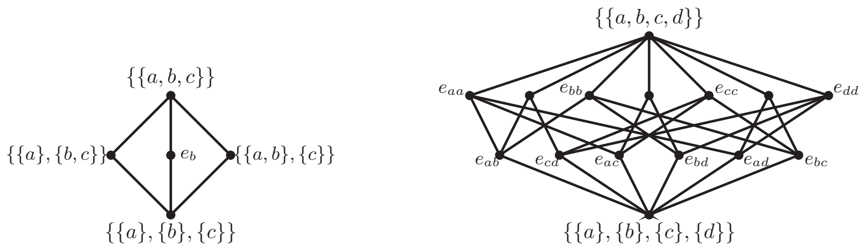


Figure 1: The lattices of equivalence relations on the three-element set $W := \{a, b, c\}$, and the four-element set $W := \{a, b, c, d\}$. In the lattice on the left, $e_b$ corresponds to the partition $\{\{b\}, \{a, c\}\}$. In the lattice on the right, $e_{xy} = \{\{x\}, \{y\}, W \setminus \{x, y\}\}$ for all $x, y \in \{a, b, c, d\}$, and the unlabelled nodes correspond, from left to right, to the partitions $\{\{a, b\}, \{c, d\}\}$, $\{\{a, c\}, \{b, d\}\}$, and $\{\{a, d\}, \{b, c\}\}$, respectively.

*Speaker.

It is well known that every lattice is a sublattice of the lattice of all equivalence relations on some set [5]. This immediately implies that the lattice logic (or the basic non-distributive logic) is sound and complete w.r.t. the class of all lattices $E(W)$ described above. Hence, in the logical framework we will discuss, we propose that the basic non-distributive logic can be regarded as the basic logic of interrogative agendas. This basic framework naturally lends itself to be enriched with various kinds of logical operators, such as epistemic operators, which represent the way in which the interrogative agenda of an agent (or a group of agents) is perceived or known by another agent (or group), and dynamic operators, which encode the changes in agents' interrogative agendas. This basic framework can be further enriched with heterogeneous operators, suitable to encode the interaction among different kinds of entities; for instance, operators that associate (groups of) agents $c$ with their (common) interrogative agenda $\Diamond c$, or operators that associate pairs $(e, \phi)$, such that $e$ is an interrogative agenda and $\phi$ is a formula, with the formula $e \rhd \phi$, representing the content of $\phi$ 'filtered through' the interrogative agenda $e$. On the basis of these ideas, a fully-fledged formal epistemic theory of the interrogative agendas of social groups and individuals can be developed.

In this talk we will discuss a semantic framework, based on *formal contexts* [3], in which multiple agents are to categorize objects based on their own views of which features are relevant.

A *formal context* is a structure $\mathbb{P} = (A, X, I)$, representing a database of objects in the set $A$, features in the set $X$, and $I \subseteq A \times X$ recording which objects have which features.[1] The epistemic attitudes of the agents who are given the task of categorizing objects in $A$ (and who might consider different subsets of $X$ as relevant for their categorization task) are modelled by associating each agent with a different element of $E(X^*)$, where $X^* := X \cup \{x^*\}$ (see footnote below). In particular, if an agent considers the features in $Y \subseteq X$ as those of relevance, this agent is associated with the element $e_Y \in E(X^*)$ which is identified (meet-generated) by the meet irreducible elements of $E(X^*)$, corresponding to the bi-partitions $\{\{x\}, X^* \setminus \{x\}\}$ for every $x \in Y$. Partitions of the form $e_Y$, $Y \subseteq X$ form a sub-lattice of $E(X^*)$. We represent the categorization performed by an agent with interrogative agenda $e_Y$ as above by the concept lattice of the formal context $\mathbb{P}_Y := (A, Y, I_Y)$ where $I_Y := I \cap (A \times Y)$.

The framework described above can be further enriched with additional relations giving rise to modal operators among different agents, agendas and categorizations, so to describe deliberation scenarios to model multi-agent interaction involving categorizations tasks such as auditing procedures.[2]

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[1] Each such $\mathbb{P}$ can be associated with $\mathbb{P}^* := (A, X^*, I^*)$, where $X^* := X \cup \{x^*\}$, and $I^* := I \cup \{(a, x^*) \mid a \in A\}$. Intuitively, $x^*$ is a redundant feature, since it does not tell apart any object from any other. Clearly, $\mathbb{P}$ and $\mathbb{P}^*$ give rise to isomorphic concept lattices.

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# Presenting quotient locales

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An advantage of the pointfree approach to topology is the ability to specify frames by generators and relations. For example, the frame of real numbers $\mathcal{O}\mathbb{R}$ has a presentation with generators $(\!(p, q)\!)$ for each $p \in \mathbb{Q} \sqcup \{-\infty\}$ and $q \in \mathbb{Q} \sqcup \{\infty\}$ subject to the following relations.

- $(\!(-\infty, \infty)\!) = 1$,
- $(\!(p, q)\!) \wedge (\!(p', q')\!) = (\!(p \vee p', q \wedge q')\!)$,
- $(\!(p, q)\!) \vee (\!(p', q')\!) = (\!(p, q')\!)$ for $p \leq p' < q \leq q'$,
- $(\!(p, q)\!) = \bigvee_{p<p'<q'<q} (\!(p', q')\!)$.

Since sublocales correspond to quotient frames it is easy to obtain presentations of sublocales of a frame by simply adding additional relations to the original presentation. For example, adding the relations $(\!(-\infty, 0)\!) = 0$ and $(\!(1, \infty)\!) = 0$ to the presentation above gives the sublocale corresponding to the closed interval $[0, 1]$.

The case of quotient locales is more subtle and it is the topic of this talk. We will describe a general procedure for obtaining presentations of open or proper locale quotients from presentations of the parent locale. The result is a relatively straightforward application of the suplattice and preframe coverage theorems [1, 3], but does not appear to have been worked out explicitly before.

An open quotient of a locale $X$ can be specified by a join-preserving closure operator on its frame of opens $\mathcal{O}X$. To present the quotient, we first ensure that the presentation for $X$ is in the form required to apply the suplattice coverage theorem of [4, 1]. A suplattice presentation for the quotient may then be found and translated back into a frame presentation. The case of proper quotients is similar and involves interior operators that are also preframe homomorphisms.

As an example, consider the description of the circle $\mathbb{T}$ as a coequaliser in **Loc**.

$$\mathbb{R} \times \mathbb{Z} \underset{+}{\overset{\pi_1}{\rightrightarrows}} \mathbb{R} \twoheadrightarrow \mathbb{T}$$

This is the coequaliser of an open equivalence relation and thus an open quotient (see [5]). The corresponding closure operator is given by the composition of the frame map $(+)^*$ and the left adjoint $(\pi_1)_!$ and sends $(\!(p, q)\!)$ to $\bigvee_{n\in\mathbb{Z}} (\!(p - n, q - n)\!)$.

Our procedure yields a presentation for $\mathcal{O}\mathbb{T}$ with the same generators $(\!(p, q)\!)$ and the following relations.

- $(\!(-\infty, \infty)\!) = 1$,
- $(\!(p, q)\!) \wedge (\!(p', q')\!) = \bigvee_{n,n'\in\mathbb{Z}} (\!((p - n) \vee (p' - n'), (q - n) \wedge (q' - n'))\!)$.
- $(\!(p, q)\!) \vee (\!(p', q')\!) = (\!(p, q')\!)$ for $p \leq p' < q \leq q'$,
- $(\!(p, q)\!) = \bigvee_{p<p'<q'<q} (\!(p', q')\!)$,

This can be shown to agree with the presentation for $\mathcal{O}\mathbb{T}$ which was worked out by hand in [2].

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# Advantages and challenges posed by PNmatrices*

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Partial non-deterministic matrices (PNmatrices) are algebraic-like structures that were introduced in the beginning of this century [1, 2, 4] as a generalization of logical matrices, by allowing the connectives to be functionally interpreted as partial multi-functions, rather than functions. Herein, we survey some recent results showing the advantages brought by taking PNmatrices, instead of logical matrices, as primary semantic structures, and also discuss the challenges raised by such a generalization.

PNmatrices allow for finite characterizations of a much wider class of logics and general recipes for various problems in logic, such as procedures to constructively update semantics when imposing new axioms [8, 7], or effectively combining semantics for two logics, capturing the effect of joining their axiomatizations [6, 13]. Whenever the underlying language is expressive enough, PNmatrices also allow for general techniques for effectively producing analytic calculi for the induced logics, over which a series of reasoning activities in a purely symbolic fashion can be performed, including proof-search and countermodel generation [16, 5, 14].

Although logics of finite PNmatrices are still decidable and in $\mathsf{coNP}$, recently, it was shown that several relevant problem known to be decidable for finite matrices become undecidable due to the incorporation of non-determinism (and partiality). Namely, given finite PNmatrices, the problems of checking if the induced logic has theorems, checking if the induced logics have the same set of theorems, or checking if the induced logics (as consequence relations) are the same are no longer decidable [9, 15].

In future research, we aim at a deeper understanding of PNmatrices, and their behaviour with respect to homomorphisms (actually, strict homomorphisms that correspond to so-called rexpansions [3]), congruences, and other basic operations and relations, extending the scope of Abstract Algebraic Logic results concerning logical matrices [10]. These are challenging questions. Gräzer [11] has shown that every multi-algebra can be obtained as a quotient of an algebra by an equivalence relation. Still, a quotient of a PNmatrix by an equivalence relation respecting its filter is still a PNmatrix, but may be defining a weaker logic. Several alternative generalizations of the traditional Leibniz operator can be explored, but it seems difficult to obtain a reasonable notion of reduced PNmatrix that may allow for a Lindenbaum-like construction based on PNmatrices. It is also unclear, even for finite matrices, how to prove (or disprove) whether a given quotient defines the same logic. These difficulties seem to be connected with the unfamiliar behaviour of abbreviations in the presence of non-determinism, as defined connectives lose their relationship with the primitive connectives with which they were defined [12], and which complicates the association of (fragments of) logics with certain clones of multi-functions.

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# Free Weak $\omega$-Categories as an Inductive Type

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## 1 Introduction

Weak $\omega$-categories were first defined by Batanin as algebras for a certain globular operad [1]. Generalising Street's computads, Batanin then defined computads for globular operads as a way to freely generate $\omega$-categories [2]. We provide an alternative, inductive definition of computads for $\omega$-categories, more suitable for computer implementation. This gives an alternative definition of $\omega$-categories, equivalent to the one of Leinster [5], as well as elementary descriptions of the universal cofibrant replacement of Garner [3] for $\omega$-categories, and a new proof of the fact that the category of computads is a presheaf topos.

## 2 Computads

Our definition of $\omega$-categorical computad is based on a distinguished collection of such objects parametrised by rooted planar trees. We adopt the name *Batanin trees* here to emphasize their interpretation as parametrizing globular pasting diagrams, and we give an inductive definitions of them and of the pasting diagrams they parametrize. We then define by induction on $n \in \mathbb{N}$, the category $\mathsf{Comp}_n$ of $n$-computads and their homomorphisms mutually inductively together with

- a forgetful functor $u_n : \mathsf{Comp}_n \to \mathsf{Comp}_{n-1}$,
- a functor $\mathrm{Cell}_n : \mathsf{Comp}_n \to \mathsf{Set}$ returning the cells of an $n$-computad
- a functor $\mathrm{Type}_n : \mathsf{Comp}_n \to \mathsf{Set}$ returning pair of parallels cells of an $n$-computad
- a transformation $\mathrm{ty}_n : \mathrm{Cell}_n \Rightarrow \mathrm{Type}_{n-1}\, u_n$, assigning a type to every cell,
- for every Batanin tree $B$, an $n$-computad $\mathrm{Pd}_n^B$ and a set $\mathrm{Full}_n(B)$ of $n$-types of $B$ that "cover" the $n$-boundary of the $B$.

An $n$-computad $C$ consists of an $(n-1)$-computad $C_{n-1}$ and a set of typed variables $V_n^C \to \mathrm{Type}_{n-1}(C_{n-1})$. Homomorphisms $C \to D$ consist of a homomorphism $C_{n-1} \to D_{n-1}$ and a function $V_n^C \to \mathrm{Cell}_n(D)$ compatible with the types. The set $\mathrm{Cell}_n(C)$ is is inductively generated by rules analogous to the term formation rules of the type theory CaTT [4]. Finally, we define a *computad* $C$ to consist of an $n$-computad $C_n$ for every $n \in \mathbb{N}$ such that $u_{n+1}C_{n+1} = C_n$.

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## 3 $\omega$-categories

The cells of a computad $C$ form a globular set $\mathrm{Cell}(C)$ and conversely every globular set $X$ gives rise to a be seen as a computad $\mathrm{Free}(X)$ where the source and target of variables are variables themselves. This defines an adjunction $\mathrm{Free} \dashv \mathrm{Cell}$ inducing a finitary monad $\mathrm{fc}^w$ on the category of globular sets. We call algebras of this monad *weak $\omega$-categories* and show that computads embed into $\omega$-categories fully faithfully. We eventually show that this notion of $\omega$-category coincides with that of Batanin and Leinster [5]

## 4 The variable-to-variable subcategory

An important class of homomorphisms of computads are the ones sending variables to variables. Such homomorphisms are closer to the ones defined for Batanin's computads [2] and they form a well-behaved lluf subcategory $\mathsf{Comp}^{\mathrm{var}}$ containing the core of $\mathsf{Comp}$ and the image of the functor Free. We then inductively construct familial representations of the functors

$$\mathrm{Cell}^{\mathrm{var}} : \mathsf{Comp}^{\mathrm{var}} \to \mathsf{Glob} \qquad\qquad \mathrm{Type}^{\mathrm{var}} : \mathsf{Comp}^{\mathrm{var}} \to \mathsf{Glob},$$

obtained by restricting the functors of cells and types to the subcategory of variable-to-variable homomorphisms. Using those representations, we show that $\mathsf{Comp}^{\mathrm{var}}$ is a presheaf topos, which famously fails for computads for strict $\omega$-categories. [6, 7].

## 5 Computadic Replacement

Garner defined a notion of universal cofibrant replacement comonad for a cofibrantly generated weak factorisation system, and gave a description of this comonad for the weak factorisation system on $\omega$-categories generated by the inclusions $\mathbb{S}^{n-1} \to \mathbb{D}^n$ of spheres into disks [3]. We shows that the free $\omega$-category on a computad $C$ is the colimit of the ones free on the $n$-computads $C_n$, and that the latter fit in pushout squares of the form

$$\begin{array}{ccc} \coprod_{v\in V_n^C} \mathbb{S}^{n-1} & \rightarrowtail & \coprod_{v\in V_n^C} \mathbb{D}^n \\ \downarrow & & \downarrow \\ C_{n-1} & \rightarrowtail & C_n \end{array}$$

In light of this theorem, we use a construction of Batanin to get a right adjoint $W : \omega\mathsf{Cat} \to \mathsf{Comp}^{\mathrm{var}}$ to the functor sending a computad to the free $\omega$-category it presents. This adjunction defines a comonad $Q$ on $\omega\mathsf{Cat}$ that coincides with the universal cofibrant replacement comonad.

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# Comparison of tabular intermediate logics

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A (propositional) *intermediate logic* is a logic being between intuitionistic logic and classical logic (see e.g. [1, 5] for needed definitions). An intermediate logic is *tabular* if it possesses a semantics given by a finite frame $P$ (here just a finite poset). In such a case the logic is denoted by $L(P)$.

We study the complexity of the following decision problem:

| |
|---|
| Problem: **Int-Log-Contain**; |
| Instance: Two finite frames $P$ and $Q$; |
| Question: Does $L(P) \subseteq L(Q)$? |

Changing the sign of containment for the sign of equality gives us the **Int-Log-Equal** problem. Here is our main result.

**Theorem 1.** *The problems* **Int-Log-Contain** *and* **Int-Log-Equal** *are* NP*-complete.*

Let us sketch the proof. A frame is *rooted* if, as a poset, it has a minimal element. The following equivalence follows from the Jankov-de Jongh theorem [2, 4] and the fact that the operations of taking generated subframes and p-morphic images preserve the satisfaction of intuitionistic formulas.

**Proposition 2.** *Let $P$ and $Q$ be finite frames. Then $L(P) \subseteq L(Q)$ iff every rooted generated subframe of $Q$ if a p-morphic image of a rooted generated subframe of $P$.*

This shows that **Int-Log-Contain** is in the NP complexity class. It also shows that the following problem is trivially reducible to **Int-Log-Contain**.

| |
|---|
| Problem: **p-Image-Gen-Sub**; |
| Instance: Two finite rooted frames $P$ and $Q$; |
| Question: Does there exist a surjective p-morfizm from a generated subframe of $P$ onto $Q$? |

We prove that the listed problems are NP-hard by presenting a polynomial-time reduction of the known NP-complete problem **Monotone-Not-All-Equal-3-Sat** [3] into **p-Image-Gen-Sub**.

Lastly, we infer that the following related problem is also NP-complete.

| |
|---|
| Problem: **p-Image**; |
| Instance: Two finite rooted frames $P$ and $Q$; |
| Question: Does there exist a surjective p-morfizm from $P$ onto $Q$? |

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# Difference–restriction algebras of partial functions with operators: discrete duality and completion

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We exhibit an adjunction between a category of abstract algebras of partial functions and a category of set quotients. The algebras are those atomic algebras representable as a collection of partial functions closed under relative complement and domain restriction; the morphisms are the complete homomorphisms. This generalises the discrete adjunction between the atomic Boolean algebras and the category of sets. We define the compatible completion of a representable algebra, and show that the monad induced by our adjunction yields the compatible completion of any atomic representable algebra. As a corollary, the adjunction restricts to a duality on the compatibly complete atomic representable algebras, generalising the discrete duality between complete atomic Boolean algebras and sets. We then extend these adjunction, duality, and completion results to atomic representable algebras equipped with arbitrary additional completely additive and compatibility preserving operators.

The paper [2] that this talk corresponds to is a sequel to *Difference–restriction algebras of partial functions: axiomatisations and representations* [1].

Collections of partial functions, together with operations on those functions (think, for example, of composition) can be studied as algebraic structures whose elements are the functions and operations are the given operations. In this framework, each distinct choice $\sigma$ of operations specifies a distinct class of algebras to be studied: we say an algebra (of the correct type) is **representable** if it is isomorphic to a collection of partial functions equipped with the operations in $\sigma$, and then we may study the class of representable algebras. And indeed many of these classes have been investigated, particularly in terms of their axiomatisability and in terms of complexity questions. (See [6, §3.2] for a guide to this literature.)

Recently, a number of categorical duality results for classes of partial function algebras have started to appear ([4, 5, 3, 7] and others), in the spirit of Stone duality between Boolean algebras and Stone spaces. This talk concerns a project aiming to develop a general unified theory for dualities of partial function algebras. In this we are guided by the example provided by the duality between Boolean algebras with operators and descriptive general frames, specifically the modular nature of that duality, where arbitrary operations satisfying certain conditions can be appended to a base duality.

For our base category—our analogue of Boolean algebras—we use the isomorphs of the following algebras.

**Definition.** An **algebra of partial functions** of the signature $\{-, \triangleright\}$ is a universal algebra $\mathfrak{A} = (A, -, \triangleright)$ where the elements of the universe $A$ are partial functions from some (common) set $X$ to some (common) set $Y$ and the interpretations of the symbols are given as follows:

- The binary operation $-$ is **relative complement**:

$$f - g := \{(x, y) \in X \times Y \mid (x, y) \in f \text{ and } (x, y) \notin g\}.$$

- The binary operation $\triangleright$ is **domain restriction**. It is the restriction of the second argument to the domain of the first; that is:

$$f \triangleright g := \{(x, y) \in X \times Y \mid x \in \mathrm{dom}(f) \text{ and } (x, y) \in g\}.$$

In [1] we axiomatised the class of representable algebras for the signature $\{-, \triangleright\}$, and we also axiomatised the smaller class of *completely representable* algebras, for the same signature. An algebra is **completely representable** if it can be embedded into an algebra of partial functions in such a way that arbitrary cardinality joins (whenever they exist) are transformed into unions. (Here, the partial order on representable algebras is defined by $a \leq b \iff a \triangleright b = a$ or equivalently $a \leq b \iff b - (b - a) = a$ and corresponds to inclusion.) The completely representable $\{-, \triangleright\}$-algebras turn out to be precisely the representable algebras that are *atomic*.

In this talk we develop duality theory for the category of completely representable $\{-, \triangleright\}$-algebras (with morphisms the complete, i.e. arbitrary join-preserving, homomorphisms). Classes of completely representable algebras are good candidates for 'discrete' dualities, that is, dualities in which the opposite category is absent any topological content. This is indeed the case here. We first exhibit an *adjunction* between the category of completely representable algebras and a category of set quotients. This generalises the discrete adjunction between the atomic Boolean algebras and the category of sets. Briefly, the adjoint of an algebra is the set of its atoms together with the quotient corresponding to the 'have the same domain' equivalence ($a \triangleright b = b$ and $b \triangleright a = a$) on partial functions.

We then define the *compatible completion* of a representable algebra, and show that the monad induced by our adjunction yields the compatible completion of any atomic representable algebra. As a corollary, the adjunction restricts to a duality on the compatibly complete atomic representable algebras, generalising the discrete duality between complete atomic Boolean algebras and sets.

Finally, we extend these adjunction, duality, and completion results to completely representable algebras equipped with arbitrary additional completely additive and compatibility preserving operators. These generalise results for atomic Boolean algebras with completely additive operators.

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# Hofmann-Mislove through the lenses of Priestley

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Let $X$ be a sober space and $L = \mathcal{O}(X)$ the frame of open subsets of $X$. The Hofmann-Mislove Theorem [7] establishes that the poset of Scott-open filters of $L$ (ordered by reverse inclusion) is isomorphic to the poset of compact saturated subsets of $X$ (ordered by inclusion). This classic result was proved in 1981 and turned out to be an extremely useful link between domain theory and topology. Several alternative proofs of the theorem have been established since then (see, e.g., [5]). Of these, the proof by Keimel and Paseka [10] is probably the most direct and widely accepted.

There is a similar result in Priestley duality for distributive lattices [11], which establishes that the poset of filters of a bounded distributive lattice $L$ is isomorphic to the poset of closed upsets of the Priestley space $X$ of $L$. A close look at the two proofs reveals striking similarities. Indeed, it was pointed out in [2] that the latter result can be obtained from the Hofmann-Mislove Theorem. This can be seen as follows:

Let $L$ be a bounded distributive lattice and $\mathcal{I}(L)$ the frame of ideals of $L$. It is well understood [9] that $\mathcal{I}(L)$ is a coherent frame, and that $\mathcal{I}$ is a functor that establishes an equivalence between the categories $\mathsf{Dist}$ of bounded distributive lattices and $\mathsf{CohFrm}$ of coherent frames. On the other hand, $\mathsf{CohFrm}$ is dually equivalent to the category $\mathsf{Spec}$ of spectral spaces [9]. Since each spectral space is sober, the Hofmann-Mislove Theorem yields that for each spectral space $X$, the poset of Scott-open filters of $\mathcal{O}(X)$ is isomorphic to the poset of compact saturated subsets of $X$. But $\mathsf{Spec}$ is isomorphic to the category $\mathsf{Pries}$ of Priestley spaces [3]. Under this isomorphism, compact saturated sets are exactly the closed upsets. Furthermore, under the equivalence between $\mathsf{CohFrm}$ and $\mathsf{Dist}$, Scott-open filters of $\mathcal{O}(X)$ correspond to filters of the distributive lattice $L$ of compact elements of $\mathcal{O}(X)$. Thus, the Hofmann-Mislove Theorem implies that the poset of filters of a bounded distributive lattice $L$ is isomorphic to the poset of closed upsets of the Priestley space $X$ of $L$.

We provide a new approach to the Hofmann-Mislove Theorem by showing that we can also go in the opposite direction and derive the Hofmann-Mislove Theorem by utilizing Priestley duality. Namely, let $L$ be a frame, and let $X$ be the Priestley space of $L$. Since every frame is a Heyting algebra, $X$ is an Esakia space [4]. Moreover, since $L$ is a complete lattice, $X$ is extremally order-disconnected. To simplify notation, we refer to extremally order-disconnected Esakia spaces simply as *localic spaces*.

For a localic space $X$, let $Y = \{x \in X \mid {\downarrow}x \text{ is clopen}\}$. By [1], if $X$ is the Priestley space of a frame $L$, then $Y$ is exactly the space of points of $L$. Thus, $L$ is spatial iff $Y$ is dense in $X$.

The key ingredient of our proof is a characterization of Scott-open filters of $L$ in the language of Priestley duality.

**Definition 1.** Let $X$ be a localic space and $C$ a closed upset of $X$. We call $C$ a *Scott-upset* if $\min C \subseteq Y$.

*Speaker.

**Theorem 2.** *Let $L$ be a frame, $X$ its Priestley space, $F$ a filter of $L$, and $C(F)$ its dual closed upset of $X$.*

(1) *$F$ is Scott-open iff $C(F)$ is a Scott-upset.*

(2) *The poset of Scott-upsets of $X$ is isomorphic to the poset of compact saturated subsets of $Y$.*

The Hofmann-Mislove Theorem is now an immediate consequence of Theorem 2. Additionally, our approach allows us to give alternate proofs for some other well-known results in domain theory and pointfree topology, including:

- Hofmann-Lawson duality between locally compact frames and locally compact sober spaces [6],
- Johnstone duality between stably locally compact frames and stably locally compact spaces [9], and
- Isbell duality between compact regular frames and compact Hausdorff spaces [8].

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# Twist-structures isomorphic to modal Nelson Lattices

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In the present study, we consider extensions of constructive logic with strong Negation by means of unary modal operations. The constructive logic with strong negation has been defined by Nelson [6] and independently by Markov [5] and can be considered as a substructural logic. Nelson lattices (N3-lattices) are an algebraic semantics for this logic. They were introduced by H. Rasiowa [7] and it is known that they form a variety. An interesting result is that every Nelson lattice can be represented as a twist-structure over a Heyting algebra. A Twist structure over a lattice is a construction used by Kalman in [4] that allows us to represent an algebra as a subalgebra of a special binary power of the lattice which is obtained by considering its direct product and its order-dual. From a result of Sendlewski we know that for every Nelson lattice $\mathbf{A}$, there exists a Heyting algebra $\mathbf{H}$ such that $\mathbf{A}$ is isomorphic to a subalgebra of a twist structure over $\mathbf{H}$. Indeed, (Sendlewski + Theorem 3.1 in [1]) given a Heyting algebra $\mathbf{H} = (H, \wedge, \vee, \to, \top, \bot)$ and a Boolean filter $F$ of $\mathbf{H}$, let

$$R(\mathbf{H}, F) := \{(x, y) \in H \times H : x \wedge y = \bot \text{ and } x \vee y \in F\}. \tag{1}$$

Then we have:

1. $\mathbf{R}(\mathbf{H}, F) = (R(\mathbf{H}, F), \wedge, \vee, *, \Rightarrow, \bot, \top)$ is a Nelson lattice, where the operations are defined as follows:
   - $(x, y) \vee (s, t) = (x \vee s, y \wedge t)$,
   - $(x, y) \wedge (s, t) = (x \wedge s, y \vee t)$,
   - $(x, y) * (s, t) = (x \wedge s, (x \to t) \wedge (s \to y))$,
   - $(x, y) \Rightarrow (s, t) = ((x \to s) \wedge (t \to y), x \wedge t)$,
   - $\top = (\top, \bot)$, $\bot = (\bot, \top)$.
2. $\neg(x, y) = (y, x)$,

Given a Nelson lattice $\mathbf{A}$, there is a Heyting algebra $\mathbf{H}_{\mathbf{A}}$, unique up to isomorphisms, and a unique Boolean filter $F_{\mathbf{A}}$ of $\mathbf{H}_A$ such that $\mathbf{A}$ is isomorphic to $\mathbf{R}(\mathbf{H}_{\mathbf{A}}, F_{\mathbf{A}})$.

In our work, we introduce an extension of the previous twist-structure construction. We consider N3-lattices endowed with unary modal operators defined as follows. A modal N3-lattice is an algebra $\langle \mathbf{A}, \blacksquare, \blacklozenge \rangle$ such that the reduct $\mathbf{A}$ is an N3-lattice and, for all $a, b \in A$,

1. $\blacklozenge a = \neg \blacksquare \neg a$,
2. if $a^2 = b^2$ then $(\blacksquare a)^2 = (\blacksquare b)^2$ and $(\blacklozenge a)^2 = (\blacklozenge b)^2$.
3. If $(a \wedge b)^2 = \bot$ then $(\blacksquare a \wedge \blacklozenge b)^2 = \bot$.

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In addition, $\mathbf{A}$ is said to be regular if it satisfies: $\blacksquare(a \wedge b) = \blacksquare a \wedge \blacksquare b$.

Furthermore, we introduce the notion of modal Heyting algebra $M\mathbf{H}$ which is an algebra $\langle \mathbf{H}, \Box, \Diamond \rangle$ such that the reduct $\mathbf{H}$ is an Heyting algebra and

$$\text{If } a \wedge b = \bot \text{ then } \Box a \wedge \Diamond b = \bot$$

The first result we want to show is that if $\mathbf{H}$ is a modal Heyting algebra and $F$ is a Boolean filter such that

$$\text{if } a \wedge b = \bot \text{ and } a \vee b \in F \text{ then } \Box a \vee \Diamond b \in F,$$

then $\mathbf{R}(\mathbf{H}, F) = (R(\mathbf{H}, F), \wedge, \vee, *, \Rightarrow, \bot, \top, \blacksquare, \blacklozenge)$ is a Modal Nelson lattice, where the operators $\Box, \Diamond$ are defined as follows:

$$\blacksquare(x, y) = (\Box x, \Diamond y), \qquad \blacklozenge(x, y) = (\Diamond x, \Box y).$$

Also, we extend the representation of Nelson lattices in terms of Heyting algebras mentioned above to the modal context. If $\mathbf{N}$ is a modal N3 lattice, then $\mathbf{H}^* = (H, \vee^*, \wedge^*, \rightarrow^*, \neg^*, 0, 1, \Box^*, \Diamond^*)$ with $H = \{a^2 : a \in N\}$, operations $a \star^* b = (a \star b)^2$ for every binary operation $\star \in N$, $\neg^* a = (\neg a)^2$, and modal operators

$$\Box^* a = (\blacksquare a)^2, \qquad \Diamond^* a = (\blacklozenge a)^2,$$

is a modal Heyting algebra. In addition, $F = \{(a \vee \neg a)^2 : a \in N\}$ is a boolean filter of $\mathbf{H}^*$ satisfying that if $a \vee^* b \in F$ and $a \wedge^* b = 0$ then $\Box^* a \vee^* \Diamond^* b \in F$. $\mathbf{N}$ is isomorphic to $\mathbf{R}(\mathbf{H}^*, F)$ as defined in (1).

In this way, we give a more general connection between modal Nelson lattices and modal Heyting algebras that the one proposed by Jansana and Rivieccio in [3] because we do not require that modal operators satisfy monotony.

From this new connection, we are able to study the directly indecomposable modal Nelson lattices and to give some results about topological duality for *MN3*. Finally, we consider the case of Nelson lattices which further satisfy

$$\text{(Prelinearity)} \quad (x \to y) \vee (y \to x) = \top$$

usually called Nilpotent Minimum algebras (see [2]). In this context, we establish the connection between Modal Nilpotent Minimum algebras and Modal Gödel algebras which are modal Heyting algebras satisfying prelinearity.

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# Transfer theorems for finitely subdirectly irreducible algebras

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The aim of this work is to determine conditions under which well-studied algebraic properties transfer from the class $\mathcal{V}_{\mathrm{FSI}}$ of finitely subdirectly irreducible members of a variety $\mathcal{V}$ to the whole variety, and, in certain cases, back again.[1] The main motivation for considering $\mathcal{V}_{\mathrm{FSI}}$ rather than the class of subdirectly irreducible members of $\mathcal{V}$ is that it is often easier to establish that certain conditions hold for this larger class. Notably, if $\mathcal{V}$ has equationally definable principal meets (a common property for varieties corresponding to non-classical logics), then $\mathcal{V}_{\mathrm{FSI}}$ is a universal class [2, Theorem 1.5]. In particular, for any variety $\mathcal{V}$ of semilinear residuated lattices, $\mathcal{V}_{\mathrm{FSI}}$ is the class of totally ordered members of $\mathcal{V}$ [3].

Recall first that a class of algebras $\mathcal{K}$ has the *congruence extension property* (for short, CEP) if for any subalgebra $\mathbf{A}$ of $\mathbf{B} \in \mathcal{K}$ and congruence $\Theta$ on $\mathbf{A}$, there exists a congruence $\Phi$ on $\mathbf{B}$ satisfying $\Phi \cap A^2 = \Theta$. We prove, generalizing [4, Theorem 3.3] (see also [8, Theorem 2.3]):

**Theorem.** A congruence-distributive variety $\mathcal{V}$ has the congruence extension property if and only if $\mathcal{V}_{\mathrm{FSI}}$ has the CEP.

When $\mathcal{V}_{\mathrm{FSI}}$ is closed under subalgebras, this result can be reformulated in terms of commutative diagrams. A class of algebras $\mathcal{K}$ is said to have the *extension property* (for short, EP) if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, embedding $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, and surjective homomorphism $\varphi_C \colon \mathbf{A} \to \mathbf{C}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B\varphi_B = \psi_C\varphi_C$, that is, the diagram in Figure 1(i) is commutative. A variety $\mathcal{V}$ has the CEP if and only if it has the EP, but this is not always the case for other classes, in particular, $\mathcal{V}_{\mathrm{FSI}}$. We show here that if $\mathcal{V}$ is a congruence-distributive variety such that $\mathcal{V}_{\mathrm{FSI}}$ is closed under subalgebras, then the EP and CEP for $\mathcal{V}$ and $\mathcal{V}_{\mathrm{FSI}}$ all coincide.

Recall next that a class of algebras $\mathcal{K}$ has the *amalgamation property* (for short, AP) if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, $\varphi_C \colon \mathbf{A} \to \mathbf{C}$, there exist a $\mathbf{D} \in \mathcal{K}$ and embeddings $\psi_B \colon \mathbf{B} \to \mathbf{D}$, $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B\varphi_B = \psi_C\varphi_C$ (see Figure 1(ii)). Let us also say that $\mathcal{K}$ has the *one-sided amalgamation property* (for short, 1AP) if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, $\varphi_C \colon \mathbf{A} \to \mathbf{C}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B\varphi_B = \psi_C\varphi_C$ (see Figure 1(iii)). It follows by [5, Lemma 2] that a variety $\mathcal{V}$ has the 1AP if and only if has the AP, but this is not always the case for other classes, in particular, $\mathcal{V}_{\mathrm{FSI}}$. We prove, generalizing [9, Theorem 9] (see also [5, Theorem 3]):

**Theorem.** If $\mathcal{V}$ has the congruence extension property and $\mathcal{V}_{\mathrm{FSI}}$ is closed under subalgebras, then $\mathcal{V}$ has the AP if and only if $\mathcal{V}_{\mathrm{FSI}}$ has the 1AP.

Finally, a class $\mathcal{K}$ of algebras has the *transferable injections property* (for short, TIP) if for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, embedding $\varphi_B \colon \mathbf{A} \to \mathbf{B}$, and homomorphism $\varphi_C \colon \mathbf{A} \to \mathbf{C}$, there exist a $\mathbf{D} \in \mathcal{K}$, a homomorphism $\psi_B \colon \mathbf{B} \to \mathbf{D}$, and an embedding $\psi_C \colon \mathbf{C} \to \mathbf{D}$ such that $\psi_B\varphi_B = \psi_C\varphi_C$ (see Figure 1(iv)). A variety has the TIP if and only if it has the CEP and AP ([1]); more generally, we show that a class of algebras that is closed under subalgebras has the TIP if and only if it has the EP and 1AP. Our previous results then yield:

[1] An algebra $\mathbf{A}$ is *finitely subdirectly irreducible* if whenever $\mathbf{A}$ is isomorphic to a subdirect product of a non-empty finite set of algebras, it is isomorphic to one of these algebras, or, equivalently, if the least element of its congruence lattice is meet-irreducible.

$$\begin{tikzcd} & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\ \mathbf{A} \arrow[ur, two heads, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\ & \mathbf{B} \arrow[ur, dashed, "\psi_B"'] & \end{tikzcd}$$

(i) EP

$$\begin{tikzcd} & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\ \mathbf{A} \arrow[ur, hook, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\ & \mathbf{B} \arrow[ur, dashed, "\psi_B"'] & \end{tikzcd}$$

(ii) AP

$$\begin{tikzcd} & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\ \mathbf{A} \arrow[ur, hook, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\ & \mathbf{B} \arrow[ur, dashed, "\psi_B"'] & \end{tikzcd}$$

(iii) 1AP

$$\begin{tikzcd} & \mathbf{C} \arrow[dr, hook, dashed, "\psi_C"] & \\ \mathbf{A} \arrow[ur, "\varphi_C"] \arrow[dr, hook, "\varphi_B"'] & & \mathbf{D} \\ & \mathbf{B} \arrow[ur, dashed, "\psi_B"'] & \end{tikzcd}$$

(iv) TIP

Figure 1: Commutative diagrams for algebraic properties

**Theorem.** A congruence-distributive variety $\mathcal{V}$ such that $\mathcal{V}_{\mathrm{FSI}}$ is closed under subalgebras has the TIP if and only if $\mathcal{V}_{\mathrm{FSI}}$ has the TIP.

Under certain conditions, these characterizations yield effective algorithms for deciding if a finitely generated variety possesses the relevant properties. Let $\mathcal{V}$ be a congruence-distributive variety that is finitely generated by a given finite set of finite algebras such that $\mathcal{V}_{\mathrm{FSI}}$ is closed under subalgebras. By Jónsson's Lemma ([6]), there exists and can be constructed a finite set $\mathcal{V}^*_{\mathrm{FSI}} \subseteq \mathcal{V}_{\mathrm{FSI}}$ of finite algebras such that each $\mathbf{A} \in \mathcal{V}_{\mathrm{FSI}}$ is isomorphic to some $\mathbf{A}^* \in \mathcal{V}^*_{\mathrm{FSI}}$. Hence it can be decided if $\mathcal{V}$ has the CEP by checking if each member of $\mathcal{V}^*_{\mathrm{FSI}}$ has the CEP. Since $\mathcal{V}$ is residually small, if $\mathcal{V}$ does not have the CEP, it cannot have the AP by [7, Corollary 2.11]. Otherwise, $\mathcal{V}$ has the CEP and it can be decided if $\mathcal{V}$ has the AP (equivalently, the TIP) by checking if $\mathcal{V}_{\mathrm{FSI}}$ has the 1AP.

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# Torsion theories and coverings of preordered groups

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A *preordered group* $(G, \leq)$ is a (not necessarily abelian) group $G = (G, +, 0)$ endowed with a preorder (i.e. a relation that is both reflexive and transitive) $\leq$ which is compatible with the addition $+$ of the group $G$: for any $a, b, c, d$ in $G$, $a \leq c$ and $b \leq d$ implies that $a + b \leq c + d$. Given two preordered groups $(G, \leq_G)$ and $(H, \leq_H)$, a morphism $f$ from $(G, \leq_G)$ to $(H, \leq_H)$ is said to be a *morphism of preordered groups* when $f: G \to H$ is a preorder preserving group morphism. All preordered groups and morphisms between them form a category, the category $\mathsf{PreOrdGrp}$ of preordered groups. The first categorical properties of $\mathsf{PreOrdGrp}$ have been studied in [4] by Clementino, Martins-Ferreira and Montoli. Among other things, they recall that the category $\mathsf{PreOrdGrp}$ of preordered groups is isomorphic to the category whose objects are pairs $(G, M)$, where $G$ is a group and $M$ a submonoid of $G$ closed under conjugation in $G$ (that is $g + m - g \in M$ for any $g \in G$ and $m \in M$), and whose arrows $f: (G, M) \to (H, N)$ are group morphisms $f: G \to H$ satisfying the condition $f(M) \subseteq N$. The submonoid $M$ in a given preordered group $(G, M)$ is called the *positive cone* of $G$ and is usually written $P_G$. Two important results of the article [4] are the fact that $\mathsf{PreOrdGrp}$ is a *normal category* [11] and that the *effective descent morphisms* in this context exactly coincide with the normal epimorphisms.

In this talk, we first present a *torsion theory* [1, 3] in the category $\mathsf{PreOrdGrp}$. This is given by the pair $(\mathsf{Grp}, \mathsf{ParOrdGrp})$ where $\mathsf{Grp}$ and $\mathsf{ParOrdGrp}$ are two full and replete subcategories of $\mathsf{PreOrdGrp}$ described as follows. The objects of $\mathsf{Grp}$ are preordered groups of the form $(G, G)$, i.e. the preordered groups whose positive cone is the entire group. Via the above mentioned isomorphism of categories, they correspond to groups $G$ endowed with the *indiscrete relation*: $a \leq b$ for any pair of elements $a$ and $b$ of $G$. The objects of $\mathsf{ParOrdGrp}$ are for their part given by *partially ordered groups*, in other words by preordered groups whose preorder is antisymmetric. Alternatively, they can also be seen as pairs $(G, P_G)$ where the positive cone $P_G$ is a *reduced monoid* (in the sense that the only element in $P_G$ having its inverse also in $P_G$ is the neutral element 0).

From this torsion theory, we directly get (thanks to the unique Proposition in [10]) the following result: $\mathsf{ParOrdGrp}$ is a *(normal epi)-reflective* subcategory, while $\mathsf{Grp}$ is *(normal mono)-coreflective* in $\mathsf{PreOrdGrp}$. In particular, the functor $F: \mathsf{PreOrdGrp} \to \mathsf{ParOrdGrp}$, associating, to any preordered group $(G, P_G)$, the partially ordered group $(G/N_G, P_G/N_G)$ (where $N_G$ is the normal subgroup of elements $x$ in $G$ such that both $x$ and $-x$ are in the positive cone $P_G$), is a *reflector*. We can prove that it has moreover *stable units* [2], hence it naturally induces a factorization system $(\mathcal{E}, \mathcal{M})$, where $\mathcal{E}$ is the class of morphisms in $\mathsf{PreOrdGrp}$ inverted by the functor $F$ and $\mathcal{M}$ the class of *trivial coverings* [2] of $\mathsf{PreOrdGrp}$. A fairly simple characterization has been obtained for this last class: a morphism $f: (G, P_G) \to (H, P_H)$ in $\mathsf{PreOrdGrp}$ is a trivial covering (i.e. is in $\mathcal{M}$) if and only if its restriction $\phi: N_G \to N_H$ to $N_G$ is an isomorphism of

*Speaker.

groups. In order to be able to describe all *coverings*, we then need to prove two intermediate results. In the proof of one of them, we build, for any preordered group $(G, P_G)$, a partially ordered group $(H, P_H)$ as well as an effective descent morphism $\pi_2 \colon (H, P_H) \to (G, P_G)$ from $(H, P_H)$ to $(G, P_G)$. Thanks to these two intermediate results, it is next possible to apply a theorem by Everaert and Gran [5] and then to get a description of coverings in the category $\mathsf{PreOrdGrp}$ of preordered groups. These are given by those morphisms in $\mathsf{PreOrdGrp}$ whose kernel is a partially ordered group. Furthermore, we also deduce that the factorization system $(\mathcal{E}', \mathcal{M}^*)$ (where $\mathcal{E}'$ is the "*stabilization*" of the class $\mathcal{E}$ and $\mathcal{M}^*$ the "*localization*" of the class $\mathcal{M}$) is *monotone-light*.

We then notice that the coverings in $\mathsf{PreOrdGrp}$ can be classified in terms of *internal actions* of the *Galois groupoid* associated with the above mentioned effective descent morphism $\pi_2$. Note that, besides its interest for the study of coverings, this last result also provides a new example of application of a theorem by Janelidze, Márki and Tholen [9] in a *non-exact* setting. Finally, we observe that, in addition to the torsion theory already exposed previously, there is also a *pretorsion theory* [6, 7] in $\mathsf{PreOrdGrp}$. The torsion-free subcategory is the same as for the torsion theory (i.e. $\mathsf{ParOrdGrp}$) while the torsion subcategory, denoted by $\mathsf{ProtoPreOrdGrp}$, is given by the full subcategory of $\mathsf{PreOrdGrp}$ whose objects are preordered groups $(G, P_G)$ for which the positive cone $P_G$ is a group. As shown in [4], the objects of $\mathsf{ProtoPreOrdGrp}$ are the so-called *protomodular objects* of $\mathsf{PreOrdGrp}$. By using the notation with the preorders, it is easily seen that these objects are actually preordered groups endowed with an equivalence relation, i.e. they are internal groups in the category of preordered sets.

All these results can be found in the article [8] written in collaboration with Marino Gran. Note that these have recently been extended to the broader context of $V$-*groups* in [12] (for a suitable *quantale* $V$).

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# First order doctrines as bipresentable 2-categories

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First order doctrines are 2-categories corresponding to different fragments of first order logic; their objects are small categories endowed with a certain structure allowing to see them as syntactic categories for first order theories, where one can interpret connectors and inference rules. Prominent examples of doctrines are **Lex**, for *left exact* categories, corresponding to cartesian logic; **Reg**, for *regular* categories, corresponding to regular logic; **Coh** for *coherent categories*, corresponding to coherent logic; but also the 2-categories **Ext** of *extensive* categories, **Adh** of *adhesive* categories, **Ex** of *exact* categories, $\mathbf{Pretop}_\omega$ of *finitary pretopoi*, or **BoolPretop** of *boolean* finitary pretopoi.

Those doctrines can also be seen as higher-dimensional versions of the different categories of propositional algebras, as $\wedge - \mathbf{Slat}$, the category of meet-semilattices, **DLat**, the category of bounded distributive lattices, **Bool** the category of boolean algebras and so on... A common feature of most of those categories of propositional algebras is that they are *finitely presentable*: they are cocomplete and generated under filtered colimits by an essentially small subcategory of compact objects, which means in some sense that arbitrary objects can be constructed from simpler ones in a nice way. Finitely presentable categories are known to enjoy a lot of excellent properties and provide a framework generalizing universal algebra, a reason for which the 1-categorical theory of presentability, as well as the more general theory of *accessibility*, have become classical topics at the intersection of category theory and model theory since [8] and [1].

The purpose of this talk, which will be based on [4], is to prove the first order doctrines aforementioned to be themselves finitely presentable in a convenient 2-dimensional sense. Previous proposal for 2-dimensional accessibility and presentability can be found in [6] and [2] in the stricter context of enriched categories. However, capturing first order doctrines as examples requires a more relaxed version involving weaker notion of filteredness and colimits: for instance, **Lex** is not 2-presentable in the sense of [6] because it has only bicolimits and not all strict ones, beside issues about its expected rank of 2-accessibility in the sense of [2].

To fix this, we introduce here relaxed notions of *bi-accessible* and *bipresentable* 2-categories and connect them to the recent advance of [3] on the theory of *flat pseudofunctors*. Our notion relies on [7] notion of *bifilteredness*, together with a convenient notion of *bicompact objects* enjoying the analog property of compact objects, against bifiltered bicolimits. We then define finitely bi-accessible categories as those having bifiltered bicolimits and an essentially small subcategory of bicompact objects generating them under bifiltered bicolimits; finitely bipresentable 2-categories are as those that are moreover bicocomplete - but similarly to the one dimensional case, this amounts to having weighted pseudolimits. We then prove that categories of flat pseudofunctors are bi-accessible - and bipresentable if their domain admits finite weighted bilimits,

*Speaker.

the latter result being part of a categorification of the well known *Gabriel-Ulmer duality*, exhibiting in some sense finitely bipresentable 2-categories as 2-categories of models of "finite bilimit 2-sketches".

Finally, we prove that the 2-category of pseudo-algebras and pseudomorphisms for a finitary pseudomonad on a finitely bipresentable 2-category is itself finitely bipresentable. This captures in particular the example of **Lex**, for its bifiltered bicolimits can be shown to be computed in **Cat**. Then, invoking the powerful paradigm of *lex colimits* introduced by [5], we prove that, for a class of finite weights $\Phi$, the corresponding 2-category of $\Phi$-*exact* categories is finitely bipresentable: but this captures all the remaining doctrines defined from exactness properties as **Reg**, **Ex**, **Coh**, **Adh**, **Ext** and $\mathbf{Pretop}_\omega$.

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# Unified inverse correspondence for DLE-logics

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Driven by the "insight that almost all completeness proofs can be reinterpreted as definability results [...] and that also correspondence theory is a kind of definability theory", Kracht [21] developed the theory of internal description, sometimes referred to as *inverse correspondence* [1]. This theory can be regarded as converse to Sahlqvist correspondence [22]; indeed, it syntactically identifies a class of first order formulas, each of which is the first order correspondent of some modal formula, and provides an effective procedure for computing such modal formula.

Goranko and Vakarelov extended Sahlqvist theory to the class of *polyadic Sahlqvist formulas* [13], also referred to as *inductive formulae* [14]. In [20], Kikot extends Kracht's result to inductive formulae, by syntactically characterizing a class of formulas in the first order language of Kripke frames for classical normal modal logic which correspond to inductive formulas in classical modal logic.

During the last decade, a line of research was developed which focuses on the order-theoretic underpinning of Sahlqvist theory, thus allowing for the generalisations of this theory from classical modal logic to wide classes of nonclassical logics. This shift from a model-theoretic to an algebraic perspective made it possible to uniformly define the class of Sahlqvist and inductive formulas/inequalities for a broad spectrum of logical languages, based on the order-theoretic properties of the algebraic interpretations of the logical connectives in each language, and to extend the algorithm SQEMA, for computing the first order correspondents of inductive formulas of classical modal logic [7], to the algorithm ALBA [8, 9], performing the same task as SQEMA for this spectrum of nonclassical languages which includes the *LE-logics*, i.e. those logics the algebraic semantics of which is given by varieties of normal/regular lattice expansions (LEs), and their expansions with fixed points [6, 3]. This very high level of generality has made it possible to extend the benefits of correspondence and canonicity results to many well known logical systems such as bi-intuitionistic (modal) logic, the Lambek-Grishin calculus [19], and the multiplicative-additive fragment of linear logic [12]. Moreover, it has also allowed for several developments and connections among the meta-theories of various logical frameworks, examples of which are a general perspective on Gödel-McKinsey-Tarski translations and correspondence/canonicity transfer results [10, 11], systematic connections among different relational semantics of a given logic [5], and systematic connections between correspondence-theoretic results and the proof-theoretic behaviour of logical frameworks [16, 17, 2, 18, 15].

While many generalizations of Sahlqvist correspondence theory have been developed in recent times, no generalizations of Kracht's theory of inverse correspondence have been investigated yet since Kikot's result. Our proposed talk presents the results of [4] which start to fill this gap, by generalizing Kikot's result from classical normal modal logic to all normal *DLE-logics*, i.e. those logics the algebraic semantics of which is given by varieties of normal distributive lattice expansions (DLEs). In particular, we introduce an *inverse correspondence* algorithm targeting inductive inequalities in any DLE-signature.

Key to this extension is the possibility to reformulate the main engine of Kracht's result in the algebraic environment of unified correspondence [6] so as to exploit the language and algorithmic tools developed there, which work across signatures and relational semantics. Indeed, to achieve this objective, we approach the problem from an exclusively order theoretic perspective by making use of a slight extension of ALBA's language and rules.

The proof-strategy adopted to achieve this result is different from Kikot's. Indeed, rather than relaxing the definition of Kracht's formula, which is given only in terms of forward-looking restricted quantifiers, we

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start by generalizing to the setting of DLE-logics the fact, well-known from classical modal logic, that inductive formulas are semantically equivalent to (a certain proper subclass) of scattered *very simple Sahlqvist* formulas in the language of *tense logic*. Accordingly, for every DLE-language $\mathcal{L}$, we syntactically characterize the class $K$ of very simple Sahlqvist $\mathcal{L}^*$-inequalities (where $\mathcal{L}^*$ is the language expansion of $\mathcal{L}$ obtained by closing the signature of $\mathcal{L}$ under the residuals of each connective in $\mathcal{L}$) which are semantically equivalent to inductive $\mathcal{L}$-inequalities. Then, we syntactically characterize the class of formulas in the ALBA-language, referred to as *Kracht's formulas* (which can be readily translated into first-order formulas of a given frame correspondence language) which target the subclass $K$, by allowing for the use of *backward-looking* restricted quantifiers. Finally, we show that each Kracht's formula in the ALBA-language can be effectively and equivalently transformed into the ALBA-output of an $\mathcal{L}^*$-inequality in $K$.

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# Modal reduction principles across relational semantics

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Previous work in the programme of unified correspondence [5, 4, 6] identified the classes of Inductive and Sahlqvist formulas for arbitrary logics that can be given algebraic semantics based on normal lattice expansions, viz. LE-logics. The members of these classes are characterised purely in terms of the order-theoretic properties of the algebraic interpretations of their connectives, and are unaffected by any change in the choice of particular semantics for the logic, as long as it is linked to the the algebraic semantics via a suitable duality. This leads to a modularization of the correspondence machinery whereby correspondents calculated uniformly by the ALBA calculus as conjunctions of set of pure quasi-inequalities. These can then be translated into first-order correspondents by applying the appropriate standard translation for the choice of dual relational semantics.

Here we approach the problem from the opposite end, by initiating a systematic comparison between the first-order correspondents of inductive formulas across different relational semantics. Some remarkable similarities between the first-order correspondents of certain well-known Sahlqvist axioms interpreted over different relational semantics have already been noted. For example, in [1] it was proven that the first-order correspondents of Sahlqvist formulas over Heyting algebra-valued Kripke frames are syntactically identical to their correspondents over ordinary Kripke frames, although the meaning is generalised (or shifted) as these formulas now belong to many-valued first-order predicate logic. In the setting of the epistemic logic of categories [2, 3] it was observed that, although formulas here denote categories rather than states of affairs, the epistemic meaning of standard axioms is arguably preserved and that moreover, the relational conditions they define (over, respectively, Kripke frames and polarity-based frames) resemble each other in very suggestive ways. For example, while the reflexivity condition defined by $p \to \Diamond p$ on Kripke frames can be expressed as $\Delta \subseteq R$ (where $\Delta$ denotes the identity relation), the same axiom imposed on polarity-based frames[1] the condition that $I \subseteq R$.

In the present work we build on these observations by building an environment in which it is possible to systematically compare the first-order correspondents of a given inductive formula across different relational semantics. Concretely, we restrict our attention to the Sahlqvist modal reduction principles (MRPs) [7] and focus on three relational settings, namely classical Kripke frames, polarity-based frames and many-valued polarity based frames. We will show that, if we write the first-order correspondents of Sahlqvist MRPs on Kripke frames in the right way, namely as inclusions of certain relational compositions, we can obtain their correspondents on polarity-based frames, roughly speaking, simply by reversing the direction of the inclusion

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[1]A polarity-based frame is a structure $(A, X, I, \mathcal{R})$ where $A$ and $X$ are non empty sets, $I \subseteq A \times X$ is the polarity relation and $\mathcal{R}$ is a family of additional relations compatible with $I$ and used to interpret modalities.

and replacing everywhere (also in compositions of relations) the identity relation $\Delta$ with the polarity relation $I$. The correctness of this procedure turns on the fact that, just like the lifting from a Kripke frame to polarity-based frame preserves its complex algebra, it also "preserves" its associated relation algebra,[2] and so relational compositions and pseudo-compositions on Kripke frames can be systematically lifted to $I$-mediated and non $I$-mediated compositions of relations on polarity-based frames. The relations $\Delta$ and $I$ thus play the role of parameters in the correspondence. This parametricity phenomenon was already observed when moving from *crisp* polarity-based frames to *many-valued* polarity-based frames. Here the relevant parameter is the truth-value algebra, which changes from the Boolean algebra $\mathbf{2}$ to an arbitrary complete Heyting algebra $\mathbf{A}$ while, syntactically, the first-order correspondents of Sahlqvist MRPs remain verbatim the same. This latter result partially generalizes that of [1] to the polarity-based setting, and provides an analogous result lifting correspondence along the dashed arrow in the following commutative diagram:

$$\begin{tikzcd}
\text{Polarity-based frames} \arrow[r] & \text{MV Polarity-based frames} \\
\text{Kripke frames} \arrow[u, hook] \arrow[r] & \text{MV Kripke frames} \arrow[u, hook, dashed]
\end{tikzcd}$$

The results presented here do not generalize smoothly beyond MRPs, as there are Sahlqvist axioms whose correspondents over polarity-based frames are not equivalent to the liftings of their correspondents on Kripke frames, e.g. $\Diamond(p \vee q) \leq \Diamond(p \wedge q)$. We conjecture that this failure is due to the loss of distributivity when moving from classical modal logic to general LE-logics and that accordingly, for general LE-logics, the present result can be generalized to all inductive inequalities which do not contain $\wedge$ of $\vee$.

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[2] By which we mean the relation algebra with constants corresponding to the identity relation and the accessibility relation together with various notions of composition on the side of Kripke frames, while on the polarity-based frame side we have constants for the incidence relation as well as the additional relations together with suitable notions of composition.

# Modal logic over semi-primal algebras

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In his generalized 'Boolean' theory of universal algebras [4] Foster introduced primal algebras. Generalizing the two-element Boolean algebra $\mathbf{2}$, an algebra $\mathbf{L}$ is *primal* if every operation on its carrier set $L$ is term-definable. During the second half of the 20th century, various weakenings of this property have been studied [9]. Since the algebras thus arising are still 'close to $\mathbf{2}$', it is reasonable to consider them as algebras of truth-values for many-valued logic. In the talk we focus on *semi-primality* [5].

**1 Definition.** A finite algebra $\mathbf{L}$ is *semi-primal* if every operation $f\colon L^n \to L$ which preserves subalgebras[1] is term-definable in $\mathbf{L}$.

In a slogan, semi-primal algebras are like primal algebras that allow proper subalgebras. Prominent examples from logic are finite Łukasiewicz chains or finite Łukasiewicz-Moisil chains. The framework of our talk is the following.

**2 Assumption.** Let $\mathbf{L}$ be a semi-primal algebra with underlying bounded lattice and let $\mathcal{A} = \mathbb{HSP}(\mathbf{L})$ be the variety it generates.

Abstractly, $\mathbf{2}$-valued coalgebraic modal logic for an endofunctor $\mathsf{T}\colon \mathsf{Set} \to \mathsf{Set}$ is summarized in the following picture based on Stone duality after 'forgetting topology':

$$\begin{tikzcd} \mathsf{Set} \arrow[loop left, "\mathsf{T}"] \arrow[r, bend left] & \mathsf{BA} \arrow[l, bend left] \arrow[loop right, "\mathsf{A}"] \end{tikzcd} \tag{1}$$

For example, if $\mathsf{T} = \mathcal{P}$ is the covariant powerset functor, then the coalgebras $\mathsf{Coalg}(\mathcal{P})$ correspond to Kripke frames and the algebras $\mathsf{Alg}(\mathsf{A})$ correspond to Boolean algebras with operator.

To relate this to our variety $\mathcal{A}$ we apply the duality for semi-primal varieties due to Keimel and Werner [7] (also see [3]) which asserts that $\mathcal{A}$ is dually equivalent to the category $\mathsf{Stone}_{\mathbf{L}}$ defined as follows

**3 Definition.** Objects of $\mathsf{Stone}_{\mathbf{L}}$ are of the form $(X, \mathbf{v})$ where $X \in \mathsf{Stone}$ and $\mathbf{v}\colon X \to \mathbb{S}(\mathbf{L})$ is continuous. Morphisms $f\colon (X,\mathbf{v}) \to (Y,\mathbf{w})$ in $\mathsf{Stone}_{\mathbf{L}}$ are continuous maps satisfying $\mathbf{w}(f(x)) \leq \mathbf{v}(x)$.

Let $\mathsf{Set}_{\mathbf{L}}$ be the category obtained from $\mathsf{Stone}_{\mathbf{L}}$ after 'forgetting topology'. There is a canonical way to lift $\mathsf{T}$ from diagram (1) to an endofunctor $\mathsf{T}'\colon \mathsf{Set}_{\mathbf{L}} \to \mathsf{Set}_{\mathbf{L}}$. We ultimately aim to describe the modal logic abstractly characterized by

$$\begin{tikzcd} \mathsf{Set}_{\mathbf{L}} \arrow[loop left, "\mathsf{T}'"] \arrow[r, bend left] & \mathcal{A} \arrow[l, bend left] \arrow[loop right, "\mathsf{A}'"] \end{tikzcd} \tag{2}$$

This also yields the more commonly investigated case

$$\begin{tikzcd} \mathsf{Set} \arrow[loop left, "\mathsf{T}"] \arrow[r, bend left] & \mathcal{A} \arrow[l, bend left] \arrow[loop right, "\mathsf{A}'"] \end{tikzcd} \tag{3}$$

obtained after composing by the forgetful functor $\mathsf{U}\colon \mathsf{Set}_{\mathbf{L}} \to \mathsf{Set}$ and its left adjoint.

[1] If $\mathbf{S}$ is a subalgebra of $\mathbf{L}$ then $a_1 \dots a_n \in S \Rightarrow f(a_1, \cdots a_n) \in S$.

**4 Example.** In our first example, let $\mathsf{T} = \mathcal{P}$. The coalgebras for the lifted functor $\mathsf{Coalg}(\mathcal{P}')$ correspond to *crisp* $\mathbf{L}$*-frames*. That is, to triples $\mathfrak{F} = (W, R, \mathbf{v})$ where $(W, R)$ is a Kripke frame and $\mathbf{v}\colon W \to \mathbb{S}(\mathbf{L})$ satisfies the compatibility condition

$$wRw' \Rightarrow \mathbf{v}(w') \subseteq \mathbf{v}(w)$$

For the $\mathbf{L}$-*models* over $\mathfrak{F}$ we only allow valuations $Val : W \times \mathsf{Prop} \to L$ which always satisfy

$$Val(w, p) \in \mathbf{v}(w).$$

In this case, diagram (2) is closely related to work by Maruyama [8]: the algebras $\mathsf{Alg}(\mathsf{A}')$ correspond to what is therein called $\mathbb{ISP}_{\mathbb{M}}(\mathbf{L})$. The non-restricted case where all valuations are allowed corresponds to diagram (3) and arises if $\mathbf{v}(w) = \mathbf{L}$ everywhere. Here, in the special case $\mathbf{L} = \mathbf{Ł}_n$ it corresponds to modal extensions of Łukasiewicz many-valued logic as described in [6].

**5 Example.** For another example, we hint at the case where $\mathsf{T} = \mathcal{L}$ is the covariant functor which generalizes $\mathcal{P}$, that is, it is defined on objects by $\mathcal{L}(X) = L^X$ and assigns to a morphism $f : X \to Y$ the morphism $\mathcal{L}f : L^X \to L^Y$ given by

$$h \mapsto (y \mapsto \bigvee\{h(x) \mid f(x) = y\}).$$

Now in (2) the coalgebras for the lifted endofunctor $\mathsf{Coalg}(\mathcal{L}')$ correspond to the $\mathbf{L}$-*labeled* $\mathbf{L}$-*frames*, that is, $(W, R, \mathbf{v})$ similar to the crisp $\mathbf{L}$-frames except that now the accessibility relation $R\colon W \to L^W$ is many-valued as well. Diagram (3) corresponds again to $\mathbf{L}$-labeled frames without further restrictions. This, in the case $\mathbf{L} = \mathbf{Ł}_n$ corresponds to the frames that have been recently investigated by algebraic means in [2] (see also [1]).

In the talk, we will report about our work in progress on the investigation of the modal logics arising from diagrams (2) and (3) in the general case, and illustrate some examples which arise by specifying to some particular functors $T$.

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# From residuated lattices to $\ell$-groups via free nuclear preimages

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Two fundamental constructions on partially ordered monoids, or *pomonoids*, are so-called *nuclear images* and *conuclear images*. A *nucleus* is closure operator $\gamma$ on a pomonoid such that $\gamma x \cdot \gamma y \leq \gamma(x \cdot y)$, while a *conucleus* is an interior operator $\sigma$ on a pomonoid such that $\sigma x \cdot \sigma y \leq \sigma(x \cdot y)$ and $\sigma \mathsf{e} = \mathsf{e}$, where $\mathsf{e}$ denotes the multiplicative unit. The image of a nucleus or a conucleus on a pomonoid (on a residuated lattice) can be equipped with the structure of a pomonoid (a residuated lattice), although not all of its operations will coincide with the corresponding operations in the original algebra.

Nuclear images allow us to construct many of the ordered algebras which arise in non-classical logic (such as pomonoids, semilattice-ordered monoids, or residuated lattices) from cancellative ones. Conuclear images then allow us to construct some of these cancellative algebras from partially ordered and lattice-ordered groups (pogroups and $\ell$-groups). In this work, we consider the problem of which algebras arise as nuclear images of conuclear images of pogroups and $\ell$-groups, and more generally as nuclear images of cancellative structures

In asking this question, we follow a line of research stemming from the classical result of Mundici [2] that MV-algebras are precisely the unit intervals of negative cones of Abelian $\ell$-groups. That is, each MV-algebra can be constructed from an Abelian $\ell$-group in two steps: first restricting to the negative cone of the $\ell$-group (consisting the elements below $\mathsf{e}$), and then further restricting to some interval $[u, \mathsf{e}]$ in this negative cone, adjusting the operations of the $\ell$-group accordingly at each step. These constructions are special cases of conuclear and nuclear images: take $\sigma x := \mathsf{e} \wedge x$ and $\gamma x := x \vee u$. Mundici's result was later extended by Galatos & Tsinakis [1] to so-called GMV-algebras, which drop the requirements of commutativity, integrality, and boundedness. These algebras are precisely the nuclear images of cancellative GMV-algebras, which in turn are precisely the kernel images of $\ell$-group, where a *kernel* is a conucleus whose image is downward closed. Galatos & Tsinakis prove this by extending Mundici's technique of good sequence to the setting of GMV-algebras. This technique, however, does not appear to apply outside the setting of GMV-algebras.

In order to extend these results beyond GMV-algebras, we first identify which pomonoids or *s$\ell$-monoids* (join-semilattice-ordered monoids) are nuclear images of cancellative pomonoids or s$\ell$-monoids, i.e. those pomonoids or s$\ell$-monoids which satisfy

$$x \cdot y \leq x \cdot z \implies y \leq z, \qquad\qquad x \cdot z \leq y \cdot z \implies x \leq y.$$

The key construction here is the *free nuclear preimage*. The nuclear image construction yields a functor from the category of *nuclear pomonoids* or *nuclear s$\ell$-monoids* (pomonoids or s$\ell$-monoids equipped with a nucleus) into the category of pomonoids or s$\ell$-monoids. The free nuclear preimage is the left adjoint of this functor. We provide an explicit description of free nuclear preimages of pomonoids and s$\ell$-monoids and use it to prove the following theorems, where a pomonoid is called *integrally closed* if it satisfies

$$x \cdot y \leq x \implies y \leq \mathsf{e}, \qquad\qquad x \cdot y \leq y \implies x \leq \mathsf{e}.$$

**Theorem 1.** *The nuclear images of (integral) [commutative] cancellative pomonoids are precisely the integrally closed (integral) [commutative] pomonoids.*

The analogous theorem for commutative s$\ell$-monoids involves what we call the *square condition*, which is a certain infinite set of equations in the signature of s$\ell$-monoids.

**Theorem 2.** *The nuclear images of [distributive] cancellative integral s$\ell$-monoids are precisely the integral s$\ell$-monoids. The nuclear images of [distributive] commutative cancellative (integral) s$\ell$-monoids are precisely the commutative integrally closed (integral) s$\ell$-monoids satisfying the square condition.*

While the subpomonoids of Abelian pogroups are precisely the commutative cancellative pomonoids, the subpomonoids of general pogroups defy any simple description. However, using a proof-theoretic argument, we can nevertheless improve the above characterization of nuclear images of cancellative pomonoids to one of nuclear images of subpomonoids of pogroups.

**Theorem 3.** *The nuclear images of (integral) subpomonoids of pogroups are precisely the integrally closed (integral) pomonoids.*

A major task which remains to be done is to extend this proof-theoretic argument from pogroups to $\ell$-groups, aiming to prove the conjecture that the nuclear images of (integral) sub-s$\ell$-monoids of $\ell$-groups are precisely the integrally closed (integral) s$\ell$-monoids.

The free nuclear preimage construction also yields a syntactic characterization of which ordered quasivarieties of pomonoids (of s$\ell$-monoids), i.e. classes axiomatized by implications between a finite set of inequalities and a single inequality, are closed under nuclear images.

**Theorem 4.** *An ordered quasivariety of pomonoids (of s$\ell$-monoids) is closed under nuclear images if and only if it is axiomatized by a set of* simple *quasi-equations, i.e. ones where in each premise $t \leq u$ the term $u$ is a variable.*

For example, the quasi-inequalities which define integrally closed pomonoids are simple, while the quasi-inequalities which define cancellative pomonoids are not.

Finally, returning to our original motivation, in the finite case we can extend these results about s$\ell$-monoids to results about residuated lattices.

**Theorem 5.** *The* finite *nuclear images of (commutative) cancellative [integral] residuated lattices are precisely the finite integral residuated lattices (satisfying the square condition).*

**Theorem 6.** *The* finite *nuclear images of conuclear images of Abelian $\ell$-groups are precisely the finite integral residuated lattices satisfying the square condition.*

One might hope to extend this argument to arbitrary $\ell$-groups, aiming to prove that the finite nuclear images of conuclear images of $\ell$-groups are precisely the finite integral residuated lattices.

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# Lifting of monotone-light factorizations

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Let $f\colon X \to Y$ be a continuous map between compact Hausdorff spaces. We say that $f$ is *monotone*, respectively *light*, if for all $y \in Y$, the fiber $f^{-1}(y)$ is connected, respectively totally disconnected. These maps were studied by [Eil34] (for metric spaces) and [Why50], where they show that every continuous map $f$ between compact Hausdorff spaces admits a factorization $f = g \circ h$ where $g$ is light and $h$ is monotone, unique up to a unique isomorphism. This is the so-called *monotone-light factorization* of compact Hausdorff spaces, which may be described as the pullback-stabilization and localization of the factorization system induced by the reflection $\pi_0\colon \mathsf{CHaus} \to \mathsf{Stn}$, a construction we make precise below, which maps each compact space $X$ to its (Stone) space $\pi_0 X$ of connected components.

Suppose $\mathcal{C}$ has finite limits. In general, a *reflection* $R\colon \mathcal{C} \to \mathcal{D}$ (a functor with a fully faithful right adjoint) merely determines a prefactorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$. Here, $\mathcal{L}$ is class of morphisms $f$ such that $Rf$ is an isomorphism. Reflections for which $(\mathcal{L}, \mathcal{R})$ is a factorization system are said to be *simple*, as defined in [CHK85]. We note that the reflection $\mathsf{CHaus} \to \mathsf{Stn}$ is simple, with $\mathcal{L}$ the class of continuous maps which induce a homeomorphism on the underlying spaces of connected components.

This relationship between reflections and prefactorizations systems was extensively studied in [CHK85]. There, some properties of reflections are shown to imply simplicity. For example, *semi-left exact* reflections (also called *admissible* in the suitable context of Janelidze-Galois theory [BJ01]) are simple, as are reflections with *stable units*.

Given a factorization system $(\mathcal{L}, \mathcal{R})$, its pullback-stabilization and localization is a pair of classes of morphisms $(\mathcal{L}_{\mathsf{stab}}, \mathcal{R}_{\mathsf{loc}})$ defined by:

$$\begin{aligned}\mathcal{L}_{\mathsf{stab}} &= \{\, f \mid p^*(f) \in \mathcal{L} \text{ for all } p \,\}\,,\\ \mathcal{R}_{\mathsf{loc}} &= \{\, f \mid \text{there exists } p \text{ of effective descent such that } p^*(f) \in \mathcal{R} \,\}\,.\end{aligned}$$

It is not always the case that $(\mathcal{L}_{\mathsf{stab}}, \mathcal{R}_{\mathsf{loc}})$ is a factorization system; when it is, we say it is the *monotone-light factorization system* induced by $(\mathcal{L}, \mathcal{R})$.

The work of [CJKP97] was centered around studying conditions for which $(\mathcal{L}_{\mathsf{stab}}, \mathcal{R}_{\mathsf{loc}})$ is a factorization system. They found in 10.3 *ibid* that semi-left exactness is not sufficient to guarantee monotone-light factorizations, and further counter-examples were later given in [Xar04]. Nevertheless, Theorem 6.9 of [CJKP97] does characterize those factorization systems for which $(\mathcal{E}_{\mathsf{stab}}, \mathcal{M}_{\mathsf{loc}})$ is a factorization system, despite the conditions given therein being difficult to verify in general.

As part of a project aiming to study categorical Galois theory for various categorical structures, we study liftings of factorization systems and of simple, semi-left exact and stable units reflections, as well as ascertaining whether lifting pullback-stable/local classes preserves stability/locality. For example, suitable factorization systems for monoidal categories induce a factorization system for the categories of the respective enriched categories, and moreover, pullback-stability is preserved.

[*] Speaker.

For instance, consider the reflection $\mathsf{Cat} \to \mathsf{Ord}$, studied in [Xar03]. In Section 2.2 ibid it is shown that this reflection is simple, and the induced factorization system $(\mathcal{L}, \mathcal{R})$ admits a monotone-light factorization $(\mathcal{L}_{\mathsf{stab}}, \mathcal{R}_{\mathsf{loc}})$, both suitable in the aforementioned sense. The reflection lifts to a (simple) reflection $\mathsf{Cat}\text{-}\mathsf{Cat} \to \mathsf{Ord}\text{-}\mathsf{Cat}$, and the induced factorization system $(\overline{\mathcal{L}}, \overline{\mathcal{R}})$ is the lifting of $(\mathcal{L}, \mathcal{R})$. The main result of [Xar22] guarantees $\overline{\mathcal{R}_{\mathsf{loc}}} = \overline{\mathcal{R}}_{\mathsf{loc}}$, a non-trivial instance where a monotone-light factorization is lifted.

As another example, consider the monoidal reflection $R \colon \Delta \to [0, \infty]^{\mathrm{op}}$ of the quantale of distribution functions into the complete real half-line. This lifts to a left-exact reflection $\hat{R} \colon \Delta\text{-}\mathsf{Cat} \to [0, \infty]^{\mathrm{op}}\text{-}\mathsf{Cat}$ of probabilistic metric spaces (see [HR13]) into Lawvere metric spaces, which induces a stable, and therefore monotone-light, factorization system. This is lifted to the factorization system induced by $\hat{R}$, also monotone-light.

These lifting results are generally achieved in two steps: by expressing the various notions of factorization systems and reflections in 2-categories with reasonable properties, and by considering pseudofunctors which preserve certain bilimits between such 2-categories. Those pseudofunctors will also preserve those notions across 2-categories, allowing us to lift factorization systems and reflections from one context to another.

This is part of on-going joint work with Maria Manuel Clementino and Fernando Lucatelli Nunes.

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# Algebraizable Weak Logics

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**Introduction** We extend the standard framework of abstract algebraic logic to the setting of logics with limited forms of substitution following recent interest in the algebraic semantics of logics based on team semantics [1, 2, 6]. Failure of closure under uniform substitution precludes us from using the standard definition of algebraizability. We give a modified definition that still preserves the uniqueness of the equivalent algebraic semantics of algebraizable logics. We also show a connection between the modified notion and classical algebraizability of the schematic fragment of a logic.

**Algebraizable Logics** The notion of algebraizable logic [3, 5] was introduced in the context of abstract algebraic logic in order to give a precise account of the relationship between logics and classes of algebras. For example, Boolean algebras correspond to propositional classical logic, and Heyting algebras correspond to intuitionistic propositional logic. The idea of algebraizability captures the equivalence between a logic and a *unique* class of algebras. A logic can have an algebraic semantics, but still fail to be algebraizable [5].

**Weak Logics** Given a signature $\mathcal{L}$, a *substitution* is an endomorphism $\sigma : \mathcal{F}m \to \mathcal{F}m$ on the term algebra $\mathcal{F}m$. A *(standard) logic* is a consequence relation $\vdash$ which is closed under *uniform substitution*, i.e. for any substitution $\sigma$, $\Gamma \vdash \phi$ entails $\sigma[\Gamma] \vdash \sigma(\phi)$. We are interested in restricting the scope of admissible substitutions. Fix a denumerable set $\mathtt{Var}$ of atomic formulas, let $\mathsf{At}(\mathcal{L})$ denote the set of all substitutions $\sigma$ such that $\sigma[\mathtt{Var}] \subseteq \mathtt{Var}$.

**Definition 1** (Weak Logic)**.** A finitary consequence relation $\vdash$ is a *weak logic* if for all substitutions $\sigma \in \mathsf{At}(\mathcal{L})$, $\Gamma \vdash \phi$ entails $\sigma[\Gamma] \vdash \sigma(\phi)$.

In order to make sense of weak logics from an algebraic perspective, we supplement a standard $\mathcal{L}$-algebra $\mathcal{A}$ with an extra predicate symbol $P$. We call the resulting structure an *expanded algebra* and refer to the interpretation $P^{\mathcal{A}}$ as $\mathsf{core}(\mathcal{A})$. Intuitively, the core of an expanded algebra captures all the elements that can be substituted for freely.

**Definition 2** (Core Semantics)**.** If $\mathbf{K}$ is a class of expanded algebras and $\Theta \cup \{\epsilon \approx \delta\}$ is a set of equations, then $\Theta \vDash^c_K \epsilon \approx \delta \iff$ for all $\mathcal{A} \in \mathbf{K}$, for all $h : \mathcal{F}m \to \mathcal{A}$ such that $h[\mathtt{Var}] \subseteq \mathsf{core}(\mathcal{A})$, if $h(x) = h(y)$ for all $x \approx y \in \Theta$, then $h(\epsilon) = h(\delta)$.

We say that an expanded algebra $\mathcal{A}$ is *core-generated* if $\mathcal{A} = \langle \mathsf{core}(\mathcal{A}) \rangle$. A quasi-variety $\mathbf{Q}$ is *core-generated* if $\mathbf{Q} = \mathbb{ISPP}_U(\mathbf{K})$ for a class of core-generated algebras $\mathbf{K}$. An expanded algebra $\mathcal{A}$ is *equationally definable* by a finite set of equations $\Sigma$ if $\mathsf{core}(\mathcal{A}) = \{x \in \mathcal{A} : \mathcal{A} \vDash \epsilon(x) \approx \delta(x)$ for all $\epsilon \approx \delta \in \Sigma\}$. A class of expanded algebras $\mathbf{K}$ is *(uniformly) equationally definable* if there is a finite set of equations $\Sigma$ such that for all $\mathcal{A} \in \mathbf{K}$, $\mathcal{A}$ is equationally definable by $\Sigma$.

*Speaker.

**Algebraizability of Weak Logics** Let $\mathcal{F}m$ and $\mathsf{Eq}$ be respectively the set of formulas and equations in $\mathcal{L}$. We define two maps, $\tau : \mathcal{F}m \to \wp(\mathsf{Eq})$ and $\Delta : \mathsf{Eq} \to \wp(\mathcal{F}m)$ (also known as *transformers* [5]), that allow us to translate formulas into equations and vice versa. We say that $\tau$ and $\Delta$ are *structural* if for all substitutions $\sigma \in \mathsf{Subst}(\mathcal{L})$, $\tau \circ \sigma = \sigma \circ \tau$ and $\sigma \circ \Delta = \Delta \circ \sigma$. For any set of formulas $\Gamma$, we let $\tau(\Gamma) := \bigcup\{\tau(\phi) : \phi \in \Gamma\}$ and for all sets of equations $\Theta$, we let $\Delta(\Theta) := \bigcup\{\Delta(\epsilon, \delta) : \epsilon \approx \delta \in \Theta\}$.

**Definition 3** (Algebraizability)**.** A weak logic $\vdash$ is *algebraizable* if there are a core-generated quasivariety $\mathbf{Q}$, equationally definable by a finite set of equations $\Sigma$, a set of equations $\tau(x)$ and a set of formulas $\Delta(x, y)$ such that:

$$\Gamma \vdash \phi \iff \tau[\Gamma] \vDash^c_{\mathbf{Q}} \tau(\phi) \qquad \phi \dashv\vdash \Delta[\tau(\phi)]$$

$$\Delta[\Theta] \vdash \Delta(\eta, \delta) \iff \Theta \vDash^c_{\mathbf{Q}} \eta \approx \delta \qquad \eta \approx \delta \equiv^c_{\mathbf{Q}} \tau[\Delta(\eta, \delta)].$$

The quasivariety $\mathbf{Q}$ is then the *equivalent algebraic semantics* of the weak logic $\vdash$. As in the standard setting, the uniqueness of equivalent algebraic semantics holds.

**Theorem 4.** *If $(\mathbf{Q}_i, \Sigma_i, \tau_i, \Delta_i)_{i \in \{0,1\}}$ witness the algebraizability of $\vdash$, then:*

$$\mathbf{Q}_0 = \mathbf{Q}_1 \qquad \Sigma_0 \equiv_{\mathbf{Q}_0} \Sigma_1 \qquad \Delta_0(x, y) \dashv\vdash \Delta_1(x, y) \qquad \tau_0(\phi) \equiv^c_{\mathbf{Q}_0} \tau_1(\phi).$$

We apply the developed framework to the systems `InqB` and `InqB`$^{\otimes}$ of classical propositional inquisitive and dependence logics to show that they are algebraizable. The intuitionistic versions `InqI` and `InqI`$^{\otimes}$, however, are not — the core is the set of join-irreducible elements, which are not equationally definable.

**Schematic Variants** Following Ciardelli [4], we define the *schematic fragment* $\mathsf{Schm}(\vdash)$ of a weak logic $\vdash$ as $\mathsf{Schm}(\vdash) := \{(\Gamma, \phi) : \forall \sigma \in \mathsf{Subst}(\mathcal{L}),\ \sigma[\Gamma] \vdash \sigma(\phi)\}$. Then $\mathsf{Schm}(\vdash)$ is a standard logic and we write $\Gamma \vdash_S \phi$ if $(\Gamma, \phi) \in \mathsf{Schm}(\vdash)$.

**Definition 5.** A weak logic $\vdash$ is *finitely representable* if there is a set of formulas $\Lambda$ such that for all $\Gamma$ and $\phi$ : $\Gamma \vdash \phi \iff \Gamma \cup \{\sigma(\phi) : \phi \in \Lambda, \sigma \in \mathsf{At}(\mathcal{L})\} \vdash_S \phi$.

Finally, we can obtain a characterisation of algebraizable weak logics in terms of representability and algebraizability of the underlying schematic fragment.

**Theorem 6.** *A weak logic $\vdash$ is algebraizable iff $\mathsf{Schm}(\vdash)$ is algebraizable and $\vdash$ is finitely representable.*

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# Internal Factorisation Systems

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In a category $\mathbb{C}$, a morphism $f$ is *orthogonal* to a morphism $g$ (written $f \downarrow g$) if for all morphisms $u$ and $v$ in $\mathbb{C}$ such that $vf = gu$, there is a unique morphism $z$ satisfying $zf = u$ and $gz = v$. A *factorisation system* on a category $\mathbb{C}$ may then be defined as a pair, $(\mathcal{E}, \mathcal{M})$, of classes of morphisms of $\mathbb{C}$ which both contain all isomorphisms of $\mathbb{C}$ and are closed under composition, such that for all $e$ in $\mathcal{E}$ and $m$ in $\mathcal{M}$, $e \downarrow m$ and for all $f$ in $\mathbb{C}$ there exist $e$ in $\mathcal{E}$ and $m$ in $\mathcal{M}$ such that $f = me$.

We internalise this notion, that is, introduce internal factorisation systems for internal categories. Firstly, for an internal category, $C$ (where $C^{\leftarrow\leftarrow}$ is defined by the pullback on the right),

$$\begin{tikzcd} C_0 \arrow[r, "e"] & C_1 \arrow[l, bend right, "d"'] \arrow[l, bend left, "c"] & C^{\leftarrow\leftarrow} \arrow[l, "m"'] \end{tikzcd} \qquad \begin{tikzcd} C^{\leftarrow\leftarrow} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "c"] \\ C_1 \arrow[r, "d"] & C_0 \end{tikzcd}$$

in a finitely complete category $\mathbb{C}$, we define the *object of points* and *object of isomorphisms* as the following pullbacks (where square brackets indicate the domain of the preceding pullback projection):

$$\begin{tikzcd} \mathrm{Pt}(C) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\leftarrow\leftarrow} \arrow[d, "m"] \\ C_0 \arrow[r, "e"] & C_1 \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Iso}(C) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Pt}(C) \arrow[d, "{\pi_1[C^{\leftarrow\leftarrow}]\pi_2[\mathrm{Pt}(C)]}"] \\ \mathrm{Pt}(C) \arrow[r, "{\pi_2[C^{\leftarrow\leftarrow}]\pi_2[\mathrm{Pt}(C)]}"] & C_1 \end{tikzcd}$$

and show that the two composites of projections $\sigma = \pi_2[C^{\leftarrow\leftarrow}]\pi_2[\mathrm{Pt}(C)]\pi_1[\mathrm{Iso}(C)] : \mathrm{Iso}(C) \to C_1$ and $\sigma' = \pi_1[C^{\leftarrow\leftarrow}]\pi_2[\mathrm{Pt}(C)]\pi_1[\mathrm{Iso}(C)] : \mathrm{Iso}(C) \to C_1$ are (equivalent) subobjects of $C_1$, which make the following left diagram a pullback (where $C^{\leftrightarrows}$ is defined as the right pullback):

$$\begin{tikzcd} \mathrm{Iso}(C) \arrow[r, "{\langle\langle\sigma',\sigma\rangle,\langle\sigma,\sigma'\rangle\rangle}"] \arrow[d, "{(d,c)\sigma}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\leftrightarrows} \arrow[d, "m\times m"] \\ C_0 \times C_0 \arrow[r, "e\times e"] & C_1 \times C_1 \end{tikzcd} \qquad \begin{tikzcd} C^{\leftrightarrows} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\leftarrow\leftarrow} \arrow[d, "{(\pi_2[C^{\leftarrow\leftarrow}],\pi_1[C^{\leftarrow\leftarrow}])}"] \\ C^{\leftarrow\leftarrow} \arrow[r, "{(\pi_1[C^{\leftarrow\leftarrow}],\pi_2[C^{\leftarrow\leftarrow}])}"] & C_1 \times C_1 \end{tikzcd}$$

For two arbitrary subobjects $\alpha : A \to C_1$ and $\beta : B \to C_1$ of $C_1$, we call the following pullback the *object of composable morphisms* (of $\alpha$ and $\beta$):

$$\begin{tikzcd} B^{\leftarrow}A^{\leftarrow} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "c\alpha"] \\ B \arrow[r, "d\beta"] & C_0 \end{tikzcd}$$

We write $A^{\leftarrow\leftarrow} = A^{\leftarrow}A^{\leftarrow}$ if $\alpha = \beta$ and define an *object of composable triples*, for three subobjects of $C_1$, similarly.

For some subobject, $\alpha : A \to C_1$ of $C_1$, $\alpha$ is *closed under composition* (in $C$) if there exists a morphism $m_\alpha : A^{\leftarrow\leftarrow} \to A$ making the following diagram commute:

$$\begin{tikzcd} A^{\leftarrow\leftarrow} \arrow[r, dashed, "m_\alpha"] \arrow[d, "\alpha\times\alpha"'] & A \arrow[d, "\alpha"] \\ C^{\leftarrow\leftarrow} \arrow[r, "m"] & C_1 \end{tikzcd}$$

Then, a pair of subobjects of $C_1$, $(\varepsilon : E \to C_1, \mu : M \to C_1)$ form an *internal factorisation system* on $C$ if $\sigma \leq \varepsilon$ and $\sigma \leq \mu$ (that is, if there exist $\sigma_\varepsilon : \mathrm{Iso}(C) \to E$ and $\sigma_\mu : \mathrm{Iso}(C) \to M$ such that $\varepsilon\sigma_\varepsilon = \sigma$ and $\mu\sigma_\mu = \sigma$), $\varepsilon$ and $\mu$ are both closed under composition, the following square is a pullback:

$$\begin{tikzcd} M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow} \arrow[r, "m(\mu\times 1)\times 1"] \arrow[d, "1\times m(1\times\varepsilon)"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1^{\leftarrow}E^{\leftarrow} \arrow[d, "m(1\times\varepsilon)"] \\ M^{\leftarrow}C_1^{\leftarrow} \arrow[r, "m(\mu\times 1)"] & C_1 \end{tikzcd}$$

and there exists a morphism $\tau : C_1 \to M^{\leftarrow}E^{\leftarrow}$ such that $m(\mu\times\varepsilon)\tau = 1_{C_1}$.

We internalise various properties of factorisation systems. Specifically, that the intersection of the classes is precisely the isomorphisms, that $\mathcal{E}$ and $\mathcal{M}$ respectively satisfy the right and left cancellation properties and that factorisations are unique up to isomorphism, which are respectively given by the fact that the following four squares are pullbacks:

$$\begin{tikzcd} \mathrm{Iso}(C) \arrow[r, "\sigma_\varepsilon"] \arrow[d, "\sigma_\mu"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\varepsilon"] \\ M \arrow[r, "\mu"] & C_1 \end{tikzcd} \qquad \begin{tikzcd} E^{\leftarrow\leftarrow} \arrow[r, "m_\varepsilon"] \arrow[d, "\varepsilon\times 1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "\varepsilon"] \\ C_1^{\leftarrow}E^{\leftarrow} \arrow[r, "m(1\times\varepsilon)"] & C_1 \end{tikzcd}$$

$$\begin{tikzcd} M^{\leftarrow\leftarrow} \arrow[r, "1\times\mu"] \arrow[d, "m_\mu"'] \arrow[dr, phantom, "\lrcorner", very near start] & M^{\leftarrow}C_1^{\leftarrow} \arrow[d, "m(\mu\times 1)"] \\ M \arrow[r, "\mu"] & C_1 \end{tikzcd} \qquad \begin{tikzcd} M^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow}E^{\leftarrow} \arrow[r, "m_\mu(1\times\sigma_\mu)\times 1"] \arrow[d, "1\times m_\varepsilon(\sigma_\varepsilon\times 1)"'] \arrow[dr, phantom, "\lrcorner", very near start] & M^{\leftarrow}E^{\leftarrow} \arrow[d, "m(\mu\times\varepsilon)"] \\ M^{\leftarrow}E^{\leftarrow} \arrow[r, "m(\mu\times\varepsilon)"] & C_1 \end{tikzcd}$$

and show that these properties are satisfied by every internal factorisation system. We then induce an order on the internal factorisation systems of an internal category and show that $\varepsilon$ and $\mu$ determine each other (up to equivalence of subobjects). We show that $(\sigma, 1_{C_1})$ forms the *trivial internal factorisation system* on an internal category, which is the top element of the order. Finally, when the base category $\mathbb{C}$ is the category **Grp**, we prove that every internal factorisation system is equivalent to the trivial one.

# Locally non-separating sublocales and Peano compactifications

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In [1], Curtis introduced the concept of a *locally non-separating* remainder in order to study the hyperspace of a non-compact space $X$. Using the property of a locally non-separating remainder, Curtis established the conditions under which a *Peano* compactification of a connected space $X$ would exist. In this talk, we discuss the analog of the concept of locally non-separating sets, in frames. We begin with a discussion of properties of sublocales, after which we define a locally non-separating sublocale and conclude by providing a generalisation for a special case of Curtis's result.

## 1 Some notes on sublocales

We recall the definition of the *supplement* and *difference* of a sublocale amongst other properties from Plewe [2]. The following results will be discussed:

**Lemma 1.1.** *If $T$ is a complemented sublocale of a frame $L$, and $S$ is any sublocale of $L$, then $S \setminus (L \setminus T) = S \cap T$.*

**Lemma 1.2.** *Let $A$ be a sublocale of $L$, and for any subset $B$ of $L$, let $\{\mathfrak{o}(b) \mid b \in B\}$ be a collection of open sublocales in $L$. Then*

$$(\bigvee_{b\in B} \mathfrak{o}(b)) \setminus A = \bigvee_{b\in B} (\mathfrak{o}(b) \setminus A).$$

**Lemma 1.3.** *$S$ is a dense sublocale of $L$ if and only if $S$ meets every non-trivial open sublocale of $L$.*

The results which follow, are concerned with useful properties of the images of sublocales under the right adjoint of a given frame homomorphism. For the purpose of this talk, $h_* : M \to L$ shall denote the right adjoint of $h : L \to M$, where $h$ is a frame homomorphism.

**Proposition 1.4.** *If $h : L \to M$ is any frame homomorphism, $a \in L$, and $T$ is a sublocale of $M$, then :*

*(1)* $h_*(T) \subseteq \uparrow a \iff T \subseteq \uparrow h(a)$,

*(2)* $h_*(T) \cap \uparrow a = \{1_L\} \iff T \cap \uparrow h(a) = \{1_M\}$,

*(3)* $h_*(T) \subseteq \mathfrak{o}(a) \iff T \subseteq \mathfrak{o}(h(a))$.

## 2 Locally non-separating sublocales

We shall assume that $L$ is a locally connected frame

**Definition 2.1.** A non-trivial sublocale $A$ of $L$ is called *locally non-separating sublocale* in $L$, if whenever $\{1\} \neq U \subseteq L$ is an open connected sublocale then $U \setminus A \neq \{1\}$ and $U \setminus A$ is connected as a sublocale.

The following proposition is required to show that every non-trivial sublocale of a locally non-separating sublocale is locally non-separating.

**Proposition 2.2.** *Let $S$ and $T$ be sublocales of $L$. $S \subseteq \overline{T}$ if and only if for every non-trivial open sublocale $\mathfrak{o}(a)$ of $L$ such that $\mathfrak{o}(a) \cap S \neq \{1\}$ then $\mathfrak{o}(a) \cap T \neq \{1\}$.*

**Proposition 2.3.** *Let $A$ and $B$ be sublocales of $L$ such that $\{1\} \neq B \subseteq A$. If $A$ is locally non-separating in $L$ then $B$ is locally non-separating in $L$.*

**Theorem 2.4.** *Let $B \subseteq L$ be a base of $L$ consisting of connected elements. Suppose $A \neq \{1\}$ is a sublocale of $L$ and that $\mathfrak{o}(b) \setminus A \neq \{1\}$ is connected for each $b \in B$. Then $A$ is locally non-separating in $L$.*

## 3 A Peano compactification with a locally non-separating remainder

Curtis established, in [1], that a connected space $X$ having a Peano compactification with a specified *locally non-separating remainder* is equivalent to the space $X$ being S-metrizable. We provide a generalisation of the above result under the assumption of $L$ being a *regular continuous* frame. In order to do so, we first define a *locally non-separating remainder* of a frame.

**Definition 3.1.** Let $S$ be a sublocale of $L$. Then $L \setminus S$ is called a *locally non-separating remainder* if $L \setminus S$ is locally non-separating in $L$.

Recall that a frame homomorphism $h : L \to M$ is said to be *open* precisely when $h_*(U)$ is an open sublocale of $L$, for every open sublocale $U$ of $M$.

**Proposition 3.2.** *If $h : L \to M$ is an onto frame homomorphism and $h_*(M)$ is an open sublocale of $L$, then $h$ is an open map.*

**Proposition 3.3.** *Let $h : L \to M$ be a compactification of $M$, where $M$ is non-compact and regular continuous, then $h_*(M) = \mathfrak{o}(a)$, if $a = \bigvee\{h_*(x) \mid x << 1_M\}$, and hence $h$ is an open map.*

The following theorem is the main result:

**Theorem 3.4.** *Suppose that $M$ is a non-compact, connected and regular continuous frame. Then $M$ has a Peano compactification $h : L \to M$ with a locally non-separating remainder $L \setminus h_*(M)$ if and only if $M$ is S-metrizable.*

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# Regular categories and soft sheaf representations

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The aim of this work is to study (soft) sheaf representations of objects of regular categories. Sheaf representations of universal algebras have been investigated since the 1970s, see e.g. [3, 5, 9, 13], inspired by several results for rings and modules obtained in the 1960s [4, 8, 12].

In particular it was observed that, for a universal algebra $A$, any distributive lattice of pairwise commuting congruences on $A$ induces a sheaf representation of $A$ [13] (i.e. a sheaf whose algebra of global sections is isomorphic to $A$). The sheaf representations over *stably compact* spaces [10] arising in this way were characterised by Gehrke and van Gool [6], who recognised the central role of the notion of softness [7]. A sheaf over a space $X$ is *soft* if, for all compact saturated[1] subsets $K \subseteq X$, every (continuous) section over $K$ can be extended to a global section. In [6], a bijection was established between (isomorphism classes of) soft sheaf representations of an algebra $A$ over a stably compact space $X$, and frame homomorphisms from the (co-compact) dual frame of $X$ to a frame of commuting congruences on $A$.

We generalise the previous result by replacing varieties of algebras—in which sheaves take values—with any *regular* category [1], i.e. a category $\mathsf{C}$ such that:

(i) $\mathsf{C}$ has finite limits.

(ii) $\mathsf{C}$ has (regular epi, mono) factorisations, i.e. every arrow $f$ in $\mathsf{C}$ can be written as $f = m \circ e$ where $e$ is a regular epimorphism and $m$ a monomorphism.

(iii) Regular epimorphisms in $\mathsf{C}$ are stable under pullbacks along any morphism.

Regular categories are a non-additive generalisation of Abelian categories. Examples of regular categories include most "algebraic-like" categories such as varieties and quasi-varieties of (possibly infinitary) algebras, any topos, the categories of Stone spaces and of compact Hausdorff spaces (and their opposite categories), and the opposite of the category of topological spaces.

If $A$ is an object of a regular category $\mathsf{C}$, the role of the "congruence lattice" of $A$ is played by the category $\mathrm{RegEpi}\, A$ of regular epimorphisms with domain $A$. Commuting congruences then correspond to *ker-commuting* objects of $\mathrm{RegEpi}\, A$ [2]. Fix an arbitrary complete lattice $P$. The functor category $[P^{\mathrm{op}}, \mathrm{RegEpi}\, A]$ can be identified with the large preorder of monotone maps $P^{\mathrm{op}} \to \mathrm{RegEpi}\, A$, with respect to the pointwise preorder. The codomain functor $\gamma\colon \mathrm{RegEpi}\, A \to \mathsf{C}$ induces a "direct image" functor

$$\gamma_*\colon [P^{\mathrm{op}}, \mathrm{RegEpi}\, A] \to [P^{\mathrm{op}}, \mathsf{C}], \quad H \mapsto \gamma \circ H.$$

Preservation of certain infima or suprema under a monotone map $H\colon P^{\mathrm{op}} \to \mathrm{RegEpi}\, A$ then corresponds to "sheaf-like" properties of the functor $\gamma_* H$. To make this precise we introduce

*Speaker.

[1] A subset of a topological space is *saturated* if it is an intersection of open sets. Whenever $X$ is locally compact and Hausdorff, "compact saturated" can be replaced with "closed" in the definition of softness.

the following notion of $\mathcal{K}$-sheaf, inspired by the work of Lurie [11, Chapter 7]. Intuitively, a $\mathcal{K}$-sheaf is a sheaf defined on the compact (or, more generally, compact saturated) subsets of a space, rather than on the open ones.

**Definition.** A $\mathsf{C}$-*valued* $\mathcal{K}$-*sheaf* on $P$ is a functor $F\colon P^{\mathrm{op}} \to \mathsf{C}$ such that:

(K1) $F(\bot_P)$ is a subterminal object of $\mathsf{C}$.

(K2) $\forall p, q \in P$, the image under $F$ of the diagram $\begin{array}{ccc} p \wedge q & \longrightarrow & p \\ \downarrow & & \downarrow \\ q & \longrightarrow & p \vee q \end{array}$ in $P$ is a pullback in $\mathsf{C}$.

(K3) $F$ preserves directed colimits.

**Theorem.** *Let* $H\colon P^{\mathrm{op}} \to \mathrm{RegEpi}\, A$ *be a monotone map whose image consists of pairwise ker-commuting elements. The following statements are equivalent:*

1. $H$ *preserves finite infima and non-empty suprema.*
2. $\gamma_* H\colon P^{\mathrm{op}} \to \mathsf{C}$ *is a* $\mathcal{K}$-*sheaf.*

Let $\mathsf{M}$ be the (large) sub-preorder of $[P^{\mathrm{op}}, \mathrm{RegEpi}\, A]$ consisting of those maps that preserve finite infima and arbitrary suprema, and whose images consist of pairwise ker-commuting elements. The previous theorem induces an isomorphism of categories between $\mathsf{M}$ and a category of soft $\mathcal{K}$-sheaf representations of $A$ over $P$ (a $\mathcal{K}$-sheaf is *soft* if all arrows in its image are regular epimorphisms). If $\mathsf{C}$ is *Barr-exact* (e.g. if $\mathsf{C}$ is a variety of algebras), regular epimorphisms with domain $A$ can be replaced with (internal) equivalence relations on $A$, and ker-commuting elements of $\mathrm{RegEpi}\, A$ with commuting equivalence relations.

Suppose that $\mathsf{C}$ is a variety of algebras (more generally, a complete and cocomplete regular category in which finite limits commute with filtered colimits). If the lattice $P$ is nice enough, the category of (soft) $\mathcal{K}$-sheaves $P^{\mathrm{op}} \to \mathsf{C}$ is equivalent to a category of ordinary (soft) sheaves. E.g., if $P$ is the lattice of compact saturated subsets of a stably compact space $X$, then $\mathcal{K}$-sheaves over $P$ correspond to ordinary sheaves on $X$. We thus recover the main result of [6].

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# Regular algebras over semimonads

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## 1 Semimonads

A monoid structure on a set $M$ induces a monad structure on the endofunctor $- \times M$ of $\mathsf{Set}$. Algebras over this monad are sets carrying a right $M$-action, meaning sets $A$ along with a map

$$A \times M \to A$$

satisfying $(am)m' = a(mm')$ and $a1 = a$. More generally, this situation occurs whenever $M$ is a monoid in a monoidal category, giving us the monad $- \otimes M$.

We can go through the same process starting with a semigroup $S$ in a monoidal category, in which case we get a semimonad $- \otimes S$, a **semimonad** meaning a monad without a unit $\eta$. Even if $S$ is made into a monoid by some unit, the category of algebras over the semimonad $- \otimes S$ is in non-trivial cases strictly larger than the category of algebras over the monad $- \otimes S$, since the latter need to be compatible with the unit ($a1 = a$ in the case of a monoid in $\mathsf{Set}$). There is, however, an easy condition which characterises semimonad algebras that are compatible with the unit.

**Proposition 1.1.** *Suppose $(T, \mu)$ is a semimonad on a category $\mathcal{C}$ and $\xi \colon TA \to A$ is a $T$-algebra. If there exists an $\eta \colon 1 \to T$ such that $(T, \mu, \eta)$ is a monad, then the following statements are equivalent:*

1. *$\xi \colon TA \to A$ is an epimorphism,*
2. *$\xi \colon TA \to A$ is a split epimorphism,*
3. *the following is a coequalizer diagram*

$$TTA \underset{T(\xi)}{\overset{\mu_A}{\rightrightarrows}} TA \xrightarrow{\xi} A\,,$$

4. *$\xi \colon TA \to A$ is an algebra over the monad $(T, \mu, \eta)$.*

## 2 Regular algebras

Consider an arbitrary semimonad $(T, \mu)$ on $\mathcal{C}$. In this case the conditions of Proposition 1.1 need not be equivalent. What kind of algebras over $(T, \mu)$ should we consider? We could use the category of all algebras over a semimonad, or we could consider the category of algebras subject to one of the conditions in Proposition 1.1.

While the properties of the category of all algebras are the easiest to describe, it is known from the $T = -\otimes S$ case that there are several properties which transfer poorly from monoids to semigroups without restricting the category of algebras to a suitable subcategory. For example, the only autoequivalences of the category of all algebras over a semimonad of the form $- \times S$ are the ones isomorphic to the identity functor, while in the monad case the autoequivalences can be much more numerous.

The category of algebras satisfying condition (3) of Proposition 1.1 appears to be a good choice. Such algebras have been studied by various authors in the $-\otimes S$ case for different choices of monoidal category, often under the name **firm** algebras (such as in [1]), although following [2], we will call such algebras **regular**. A large part of our results come from translating results about $-\otimes S$ into the general semimonad setting. This is quite straightforward in the situation that we will describe next.

# 3 Adjoint semimonads

If we want to prove things about the category of regular algebras of a semimonad $T$, it would be very helpful if $T$ preserved coequalizers. In the case of semimonads of the form $-\otimes S$, this condition is naturally satisfied by assuming that the ambient monoidal category is closed. Furthermore, in that case the semimonad $-\otimes S$ will have right adjoint.

Motivated by this, we consider adjoint semimonads $L$, meaning semimonads $(L, \mu)$ such that $L$ has a right adjoint $R$. It is well known that the right adjoint $R$ of a semimonad will carry a cosemimonad structure and that the category of coalgebras over $R$ will be isomorphic to the category of algebras over $L$ via the correspondence

$$L(A) \to A \quad \Leftrightarrow \quad A \to R(A)\,.$$

However, this isomorphism between algebras and coalgebras need not be compatible with the notion of regularity. A coalgebra corresponding to a regular algebra need not satisfy the dual of the regularity condition. However, if it does, we say that the algebra (and the corresponding coalgebra) is **coregular**.

This gives us another class of algebras to consider, but for certain semimonads the category of regular algebras is equivalent to the category of coregular algebras. For example, we can achieve this by assuming things about the coreflection functor of algebras into regular algebras, such as the coreflector acting by taking the coequalizer of the pair

$$LLA \underset{L(\xi)}{\overset{\mu_A}{\rightrightarrows}} LA\,.$$

Finally, in this context, the epimorphism conditions of Proposition 1.1 are of renewed interest, since under suitable assumptions the category of regular algebras is equivalent to the category of algebras such that $L(A) \to A$ is a nice epimorphism and the corresponding $A \to R(A)$ is a nice monomorphism. This assumption on algebras is simple and self-dual, which makes it an appealing alternative to regularity.

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# Remarks on enriched protomodularity

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This work was motivated by the study of the categories $\mathsf{OrdGp}$ of preordered groups and $\mathsf{OrdAb}$ of preordered abelian groups, enriched in the category $\mathsf{Ord}$ of preordered sets. Note that $\mathsf{OrdGp}$ differs from the category of the internal groups in $\mathsf{Ord}$, since the inversion morphism of the group structure is not necessarily monotone. As a consequence, many of the nice algebraic properties of (abelian) groups fail to hold in that context.

In this talk we focus on the algebraic property of protomodularity, that is, on the validity of the Split Short Five Lemma, and on a possible enriched version of it. Although the category of (abelian) groups is protomodular, $\mathsf{OrdGp}$ and $\mathsf{OrdAb}$ are not. It is as if the preorder structure works against protomodularity. However, the enriched preordered structure on morphisms does work in favour of protomodularity in the following sense. Having in mind the role of comma objects in the enriched context, we consider some of the characteristic properties of protomodularity with respect to comma objects instead of pullbacks. We show that the equivalence between protomodularity and certain properties on pullbacks also holds when replacing conveniently pullbacks by comma objects in any finitely complete category enriched in $\mathsf{Ord}$. We show that $\mathsf{OrdAb}$ gives an example of such an enriched protomodular category.

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# Modality in worlds with different logics

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Most non-classical logics can be understood in a modal structure with classical worlds (e.g. Priest's approach in [1]). Whats more, we can also make use of the Kripke style semantics to study modality independently of the assumption of a background logic – i.e. the logic regulating the non-modal connectives. However little is said about modal structures in which there are worlds operating in different logics (see. [2]). An important concern is whether or not it makes sense to evaluate the possibility of a statement in a world $w$ by inspecting its truth value in a world $w'$ in a alternative logic. As argued by Kripke in [3], when we are navigating through worlds, the referents for names should be fixed in order to make sense of intuitive distinctions between counterfactuals (e.g. "Newton could have died young" and "the inventor of calculus could have died young"). Likewise, we argue that something must be preserved when we change the logical background – for the validity of a formula can have different meanings when evaluated in different logics.

We address this issue by introducing a modal notion and structure that accommodate communication between logic systems by fixing a common lattice $L$ where different logics build their semantics (see. [4]). We suggest that from a collection of logics with complete lattice semantics $\Sigma$, one should build a common lattice $L$ (which always exist) that has $\Sigma$ as a collection of complete sublattices. The common order offered in $L$ can then be taken as the background where the appropriate communication of logical values occur. Necessity and possibility of a statement will not solely rely on the satisfaction relation in each world and the accessibility relation. Instead, the value of a formula $\Box\varphi$ will be defined in terms of a comparison between the values of $\varphi$ in accessible worlds and the common lattice $L$. This is done by relativizing each value of $\varphi$ in an accessed world $w'$ to a value in the current world $w$ using the **down-interpretation** or the **up-interpretation**:

**Definition 1.** *In a base lattice $L$, a value $a \in L$ is interpreted in a sublattice $L'$ as:*

1. ***Down-interpretation*** *- the least value in $L'$ that is larger than* ***all values of*** *$L'$* ***that are smaller than*** *$a$ – formally, $a^{L'} = \bigcup_{L'}\{x \in L' \mid x \leq a\}$.*

2. ***Up-interpretation*** *- the largest value in $L'$ that is smaller than* ***all values of*** *$L'$* ***that are bigger than*** *$a$ – formally, $a^{L'} = \bigcap_{L'}\{x \in L' \mid x \geq a\}$.*

With this natural interpretation of multi-logic modal scenarios, we will show a series of cases where a formula $\varphi$ can be said to be necessary/possible even though an/all accessible world/s falsify $\varphi$. This possibility arises from natural algebraic properties of sublattices.

Subsequently, we will characterize these modal structures by establishing conditions that imply validity of the axiom K and/or the rule of necessitation. We shall present this analysis in a limited setting, where we fix a unary function $(-)$ over the common lattice $L$ for negation

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in the common lattice and interpret disjunction and conjunction as the joint $(+)$ and meet operations $(.)$. With implication $(x \to y)$ defined as $(-x + y)$ and a set of designated values $F \subseteq L$ defining validity, we obtain general conditions for a $L'$-world $w$ to satisfy axiom $K$. We also define a semantical notion of necessitation restricted to worlds of a given sublattice $L^*$, showing the conditions over $L^*$ and $F$ required for this necessitation to hold.

We should further characterize the many logic modal structures with respect to the finite model property (FMP). This is an important property of modal systems and it is intimately related to decidability of modal logics. We say that a semantic has the FMP if, for every non valid formula, there is a structure with finite number of worlds that falsify the formula. Lattices being of varying complexities can produce scenarios where FMP fails. We will nevertheless show interesting classes of infinite lattices that still produce structures that have FMP.

Expanding the universe of modal structures to varying logics operating in each worlds opens up the possibility of investigating more nuanced notions of frames. Traditional modal theory define frames solely with respect to the accessibility relation between worlds. Now, not only we can say that the accessibility relation is for instance transitive, but that it also relates worlds with some lattices. Defining a relation of 'more classical' between sublattice of an $L$, we will produce frames where the accessibility relation goes from less classical worlds to more classical worlds. We will show some examples of this phenomena where the base lattice is produced by the twist of boolean algebras as proposed by Fidel in [5] and Vakarelov in [6].

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# An algebraic theory of clones

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Clones are sets of finitary operations on a fixed carrier set that contain all projections and are closed under composition. They play an important role in universal algebra, since the set of all term operations of an algebra always forms a clone and in fact every clone is of this form. Clones play another important role in the study of first-order structures. Indeed, the polymorphism clone of a first-order structure, consisting of all finitary functions which preserve the structure, forms a clone. Polymorphism clones carry information about the structures that induce them, and are a powerful tool in their analysis. Clones are also important in theoretical computer science. Many computational problems can be phrased as constraint satisfaction problems (CSP). If we fix a structure $A$, the problem $\mathrm{CSP}(A)$ is the computational problem of deciding whether a given conjunction of atomic formulas over the signature of $A$ is satisfiable in $A$. The seminal discovery in the algebraic approach to CSP is Jeavons's result of [4] that, for a finite structure $A$, the complexity of $\mathrm{CSP}(A)$ is determined by the polymorphism clone of $A$.

A one-sorted algebraic theory of clones has recently been introduced in [2]. Indeed, clone algebras (CA) form a variety of algebras in the universal algebraic sense. A crucial feature of this approach is connected with the role played by variables in free algebras and projections in clones. In clone algebras these are abstracted out, and take the form of a countable infinite system of fundamental elements (nullary operations) $\mathsf{e}_1, \mathsf{e}_2, \dots, \mathsf{e}_n, \dots$ of the algebra. One important consequence of the abstraction of variables and projections is the abstraction of term-for-variable substitution and functional composition in CAs, obtained by introducing an $(n+1)$-ary operator $q_n$ for every $n \geq 0$. Roughly speaking, $q_n(a, b_1, \dots, b_n)$ represents the substitution of $b_i$ for $\mathsf{e}_i$ into $a$ for $1 \leq i \leq n$ (or the composition of $a$ with $b_1, \dots, b_n$ in the first $n$ coordinates of $a$).

In [2] the authors have shown that the finite-dimensional clone algebras generate the variety of clone algebras and are the abstract counterpart of the clones of finitary operations, where the dimension of an element in a clone algebra is an abstraction of the notion of arity. In [2] it was also given an answer to the lattice of equational theories problem proposed by Birkhoff and Maltsev: a lattice is isomorphic to a lattice of equational theories (of finitary algebras) if and only if it is isomorphic to the lattice of all congruences of a finite-dimensional clone algebra.

The most natural CAs, the ones the axioms are intended to characterise, are algebras of functions, called *functional clone algebras* (FCAs). The elements of a FCA with *value domain A* are infinitary operations, called here t-operations. They are functions $\varphi : \mathsf{a} \to A$, whose domain $\mathsf{a}$, called a *trace on A*, is a nonempty subset of $A^\omega$ satisfying the following condition: $\forall r, s \in A^\omega .\ s \in \mathsf{a}$ and $|\{i : s_i \neq r_i\}| < \omega \Rightarrow r \in \mathsf{a}$. A trace $\mathsf{a}$ on $A$ is *complete* if $\mathsf{a} = A^\omega$ and it is *basic* if it is minimal. In this framework the nullary operators are the projections $p_i$, defined by $p_i(s) = s_i$ for every $s \in \mathsf{a}$, and $q_n(\varphi, \psi_1, \dots, \psi_n)$ represents the $n$-ary composition of $\varphi$ with $\psi_1, \dots, \psi_n$, acting on the first $n$ coordinates: $q_n(\varphi, \psi_1, \dots, \psi_n)(s) = \varphi(\psi_1(s), \dots, \psi_n(s), s_{n+1}, s_{n+2}, \dots)$, for every $s \in \mathsf{a}$. Every clone algebra $\mathcal{C}$ of universe $C$ is isomorphic to a FCA with value domain $C$, whose trace is basic and contains the sequence $(\mathsf{e}_1^{\mathcal{C}}, \mathsf{e}_2^{\mathcal{C}}, \dots, \mathsf{e}_n^{\mathcal{C}}, \dots)$.

The most part of clone algebras are not finite-dimensional. Then it is natural to investigate what are the algebraic structures that correspond to clone algebras in full generality. We have introduced in [5] a new general framework for algebras and clones, called *universal clone* *algebra*. To make a comparison, algebras and clones of finitary operations are to universal algebra what t-algebras and clone algebras are to universal clone algebra. A t-algebra is a tuple $\mathbf{A} = (A, \mathsf{a}, \sigma^{\mathbf{A}})_{\sigma\in\tau}$, where $\mathsf{a}$ is a trace on $A$ and $\sigma^{\mathbf{A}} : \mathsf{a} \to A$ is a t-operation for every $\sigma \in \tau$.

We have two algebraic levels: the lower degree of t-algebras and the higher degree of clone algebras. There are many ways to move between these levels. If $K$ is a class of t-algebras, then $K^{\uparrow}$ is a class of clone algebras. If $H$ is a class of clone algebras, we have two ways to go down: $H^{\downarrow}$ and $H^{\Downarrow}$ are two classes of t-algebras such that $H^{\downarrow} \subseteq H^{\Downarrow}$. After generalising the usual algebraic construction to t-algebras (namely, t-subalgebra, t-product, t-homomorphic image, t-expansion and t-variety), we prove that (1) If $K$ is a t-variety of t-algebras, then $K^{\uparrow}$ is a variety of clone algebras; (2) If $H$ is a variety of clone algebras, then $H^{\downarrow}$ is a t-variety and $H^{\Downarrow}$ is a t-variety closed under t-expansion (*Et-variety*, for short). We provide concrete examples that general results in universal clone algebra, when translated in terms of algebras and clones, give new versions of known theorems in universal algebra.

**Theorem 0.1.** (Birkhoff Theorem for t-algebras) *Let $K$ be a class of t-algebras of the same type. Then the following conditions are equivalent: (1) $K$ is an Et-variety of t-algebras; (2) $K$ is an equational class of t-algebras; (3) $K^{\uparrow}$ is a variety of clone algebras and $K = K^{\uparrow\Downarrow}$.*

**Theorem 0.2.** (Birkhoff Theorem for algebras) *Let $H$ be a class of algebras of the same type and $H^{\star}$ be the class of all t-algebras obtained by gluing together algebras in $H$ (formally defined in [5]). Then the following conditions are equivalent: (1) $H$ is a variety of algebras; (2) $H$ is an equational class of algebras; (3) $H^{\star}$ is an Et-variety of t-algebras; (4) $(H^{\star})^{\uparrow}$ is a variety of clone algebras and $H^{\star} = (H^{\star})^{\uparrow\Downarrow}$.*

The study of topological variants of Birkhoff's theorem was initiated by Bodirsky and Pinsker [1] for locally oligomorphic algebras, and generalised recently by Schneider [6] and Gehrke-Pinsker [3]. These authors provide a Birkhoff-type characterisation of all those members $\mathbf{T}$ of the variety HSP($\mathbf{S}$) generated by a given algebra $\mathbf{S}$, for which the natural homomorphism from Clo$\mathbf{S}$ onto Clo$\mathbf{T}$ is uniformly continuous with respect to the uniformity of pointwise convergence.

If $\mathbf{A}$ is a t-algebra, then $\mathbf{A}^{\uparrow}$ is the term clone algebra over $\mathbf{A}$, the t-algebra analogue of the term clone of an algebra.

**Theorem 0.3.** (Topological Birkhoff for t-algebras) *Let $\mathbf{A}, \mathbf{B}$ be t-algebras of the same type and let $\mathsf{b}$ be the trace of $\mathbf{B}$. Then the following are equivalent: (1) $\mathbf{B}$ is an element of the Et-variety generated by $\mathbf{A}$, and the natural homomorphism from the term clone algebra $\mathbf{A}^{\uparrow}$ onto the term clone algebra $\mathbf{B}^{\uparrow}$ is uniformly continuous. (2) Every t-subalgebra $\mathbf{B}_s$ of $\mathbf{B}$ generated by $s \in \mathsf{b}$ is a t-homomorphic image of a t-subalgebra of a finite t-power of $\mathbf{A}$.*

We remark that the t-subalgebras involved in (2) depend on the trace $\mathsf{b}$. For example, if $s \in \mathsf{b}$ and $|\{s_i : i \in \omega\}| = \omega$, then $\mathbf{B}_s$ is not in general finitely generated. As a corollary, besides the version of topological Birkhoff by Schneider [6] and Gehrke-Pinsker [3], we get new versions of the topological Birkhoff's theorem for algebras depending on the choice of the trace $\mathsf{b}$.

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# On relative principal congruences in term quasivarieties

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In this work we introduce and study term quasivarieties. Roughly speaking, they are quasivarieties in which there are some binary terms characterizing relative principal congruences. As application we study relative compatible functions in this kind of quasivarieties.

Let $A$ be an algebra. As usual, $\mathrm{Con}(A)$ denotes the partially ordered set of all congruences on $A$ with respect to the inclusion. We write $\theta(a,b)$ for the smallest congruence which contains the pair $(a,b)$: these congruences are called principal congruences. Given a quasivariety $\mathrm{K}$ and $A \in \mathrm{K}$ it is natural to study only those congruences of $A$ whose quotient $A/\theta$ belongs to $\mathrm{K}$. If $\theta \in \mathrm{Con}(A)$, we say that $\theta$ is a K-congruence if $A/\theta \in \mathrm{K}$. Let K be a quasivariety and $A \in \mathrm{K}$. We write $\mathrm{Con}_{\mathrm{K}}(A)$ for the partially ordered set of all K-congruences on $A$ with respect to the inclusion. We write $\theta_{\mathrm{K}}(a,b)$ for the smallest K-congruence containing the pair $(a,b)$: these congruences are called principal K-congruences (or relative principal congruences for short). Note that if K is a variety and $A \in \mathrm{K}$, then $\mathrm{Con}(A) = \mathrm{Con}_{\mathrm{K}}(A)$, so $\theta_{\mathrm{K}}(a,b) = \theta(a,b)$.

Given an algebra $A$, a function $f : A^n \to A$ is said to be compatible if any congruence of $A$ is a congruence of the algebra $(A,f)$. If K is a quasivariety and $A \in \mathrm{K}$, we say that $f$ is a K-compatible operation of $A$ if any K-congruence of $A$ is a K-congruence of $(A,f)$, or, equivalently, for every $\theta \in \mathrm{Con}_{\mathrm{K}}(A)$ and $a_1, b_1, \ldots, a_n, b_n \in A$ the following condition is satisfied: if $(a_i, b_i) \in \theta$ for $i = 1, \ldots n$, then $(f(a_1,\ldots,a_n), f(b_1,\ldots,b_n)) \in \theta$ (note that if K is a variety, $f$ is K-compatible if and only if $f$ is compatible). The logical motivation for the study of K-compatible operations comes from the notion of implicit connectives in algebraizable logics: in [3] Caicedo established a link between the implicit connectives of an algebraizable logic and the relatively compatible functions of its corresponding quasivariety obtained via the process of algebraization of Blok-Pigozzi [1].

Let $A$ be an algebra, $f : A^n \to A$ a function and $\hat{a} = (a_1,\ldots,a_n) \in A^n$. For $i = 1,\ldots,n$ we define unary functions $f_i^{\hat{a}} : A \to A$ by $f_i^{\hat{a}}(b) := f(a_1,\ldots,a_{i-1},b,a_{i+1},\ldots,a_n)$. Let K be a quasivariety. There is a link between the principal K-congruences and the K-compatibility of $f$. More precisely, $f$ is K-compatible if and only if for every $\hat{a} \in A^n$, $x, y \in A$ and $i = 1,\ldots,n$, $(f_i^{\hat{a}}(x), f_i^{\hat{a}}(y)) \in \theta_{\mathrm{K}}(x,y)$. Hence, a good description of the principal K-congruences may be a useful tool for the study of K-principal operations and its possible applications. If there is no ambiguity, we write relatively compatible operation instead of K-compatible operation.

In this work we are interested in quasivarieties with some particular properties. Let K be a quasivariety. We say that K is a term quasivariety if there exist an operation of arity zero $e$ and a family of binary terms $\{t_i\}_{i \in I}$ such that for every $A \in \mathrm{K}$, $\theta \in \mathrm{Con}_{\mathrm{K}}(A)$ and $a, b \in A$ the following condition is satisfied: $(a,b) \in \theta$ if and only if $(t_i(a,b), e) \in \theta$ for every $i \in I$. In such case we say that $(e, \{t_i\}_{i \in I})$ is a pair associated to K. If a term quasivariety K is a variety, then we also say that K is a term variety.

The definition of term quasivariety is motivated by the fact that there are many quasivarieties for which the procedure to obtain a description of the relative principal congruences is exactly the same. For instance, the variety of Heyting algebras is a term variety. Indeed, if $(A, \wedge, \vee, \to, 0, 1)$ is a Heyting algebra, $\theta \in \mathrm{Con}(A)$ and $a, b \in A$, then $(a,b) \in \theta$ if and only if $((a \to b) \wedge (b \to a), 1) \in \theta$.

Let K be a a term quasivariety and $(e, \{t_i\}_{i \in I})$ a pair associated to K. For every $A \in \mathrm{K}$ we define $\Sigma = \{e/\theta : \theta \in \mathrm{Con}_{\mathrm{K}}(A)\}$. Note that $\Sigma$ is a poset with the order giving by the inclusion. Moreover, $\Sigma$ is a complete lattice. Let $X \subseteq A$. We define $\langle X \rangle = \bigcap_{X \subseteq e/\theta}(e/\theta)$, which is the smallest element of $\Sigma$ containing $X$.

The main goal of this work is to describe the relative principal congruences in term quasivarieties (we also show that there exist quasivarieties which are not term quasivarieties). More precisely, we show that if K is a term quasivariety and $(e, \{t_i\}_{i \in I})$ is a pair associated to K, then for every $A \in \mathrm{K}$ and $a, b, x, y \in A$ the following condition is satisfied:

$$(x, y) \in \theta_{\mathrm{K}}(a, b) \text{ if and only if } t_j(x, y) \in \langle \{t_i(a, b)\}_{i \in I} \rangle \text{ for every } j \in I.$$

We use this description in order to characterize K-compatible functions and we give two applications of this property: 1) we give necessary conditions on K for which for every $A \in \mathrm{K}$ the K-compatible functions on $A$ coincides with a polynomial over finite subsets of $A$; 2) we give a method to build up K-compatible functions. Finally, we apply the above mentioned results in order to obtain known properties about relative principal congruences and relatively compatible operations in many quasivarieties of interest for algebraic logic, as for example hemi-implicative semilattices [4, 12], RWH-algebras [5], subresiduated lattices [5, 7], semi-Heyting algebras [11], implicative semilattices [10], commutative residuated lattices [8, 9], BCK-algebras [2] and Hilbert algebras [6].

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# Unitless Frobenius quantales*

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It is often stated that a Frobenius quantale necessarily is unital. While this is correct if Frobenius quantales are defined starting from a dualizing element, it is also possible to consider negations as primitive operations and axiomatize them so to ensure some coherency w.r.t. implications.

**Definition 1.** A *Frobenius quantale* is a tuple $(Q, *, {}^{\perp}(-), (-)^{\perp})$ where $(Q, *)$ is a quantale and ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q$ are inverse antitone maps satisfying

$$x\backslash {}^{\perp}y = x^{\perp}/y\,, \qquad \text{for all } x, y \in Q. \tag{1}$$

The map $(-)^{\perp}$ is called the *right negation* while the map ${}^{\perp}(-)$ the *left negation*. A *Girard quantale* is a Frobenius quantale for which right and left negations coincide.

Axiom (1) explicitly appears in [4] and similar (and actually equivalent) relations, such as

$$x\backslash y = x^{\perp}/y^{\perp}\,, \qquad x/y = {}^{\perp}x\backslash {}^{\perp}y\,, \qquad {}^{\perp}x\backslash y = x/y^{\perp}\,.$$

have been pointed out in the literature, see e.g. [2, 9]. Of course, if a quantale $Q$ has a dualizing element 0, then the two negations ${}^{\perp}(-) := 0/-$ and $(-)^{\perp} := -\backslash 0$ satisfy (1). Also, if a Frobenius quantale $Q$ is unital, then the two negations give rise to a dualizing element $1^{\perp} = {}^{\perp}1$, so the previous definition does not yield novelties for unital quantales. According to it, however, we can have Frobenius quantales that are unitless. For example, for a quantale $Q$, its Chu construction $Chu(Q)$ is a Girard quantale which is unital if and only if $Q$ is unital.

Our aim is to have a first glance on these structures and decide on the worthiness of future research. We firstly observe that the standard representation theory via phase quantales can be lifted to unitless Girard quantales and even to unitless Frobenius quantales.

**Definition 2.** For a quantale $Q$, a *Serre*[1] *Galois connection* is a Galois connection on $(l, r)$ on $Q$ such that $l \circ r = r \circ l$ and $x\backslash l(y) = r(x)/y$, for all $x, y \in Q$.

**Theorem 3.** *If $(l, r)$ is a Serre Galois connection on $Q$, then $j = r \circ l = l \circ r$ is a nucleus on $Q$. The quantale of fixed-points of $j$, $Q_j$, is then a Frobenius quantale where the left (resp., right) negation is given by the restriction of $l$ (resp., $r$) to $Q_j$.*

Every Frobenius quantale arises in this way:

**Theorem 4.** *If $Q$ is a Frobenius quantale, then the powerset quantale $P(Q)$ has a canonical Serre Galois connection $l, r$ such that, for $j = l \circ r$, the quantale $P(Q)_j$ is isomorphic to $Q$.*

Motivations and examples for developing this theory stem from the following result:

**Theorem 5** (See e.g. [7, 2, 3, 11, 10]). *The quantale of sup-preserving endomaps of a complete lattice $L$ is a Frobenius quantale if and only if $L$ is completely distributive.*

*Full version available as [1].

†Speaker.

[1] The naming originates from [9].

and also from lattice theoretic constructions [12, 5] related to Raney's notion of tight Galois connection [8]. Recall the definition of Raney's transforms:

$$f^{\vee}(x) = \bigvee_{x \not\leq t} f(t)\,, \qquad\qquad g^{\wedge}(x) = \bigwedge_{t \not\leq x} g(t)\,.$$

For $L$ a complete lattice, a sup-preserving preserving map $f : L \longrightarrow L$ is *tight* if $f = f^{\wedge\vee}$. We decompose the sufficient condition of Theorem 5 as follows:

**Theorem 6.** *The set of tight endomaps of a complete lattice $L$ is a Girard quantale.*

Then, using Raney's characterisation of completely distributive lattices [8], we have:

**Theorem 7.** *The Girard quantale of tight endomaps of $L$ is unital if and only if $L$ is a completely distributive lattice, if and only if the identity of $L$ is tight, if and only if every sup-preserving endomap of $L$ is tight.*

There is a precise analogy between tight maps and trace class operators on an infinite dimensional Hilbert space $H$: these are nuclear maps [6] in the appropriate autonomous categories. Let $B_1(H)$ be the ideal of trace class operators: as an algebra, it cannot have a unit. The trace operation allows to define a (self-adjoint) Serre Galois connection $(l, l)$ on the powerset quantale $P(B_1(H))$, where $B_1(H)$ is considered as a monoid w.r.t. multiplication. Letting $j = l^2$ in the next statement, we obtain a generalised version of the Girard quantale of subspaces of a finite dimensional C$^*$-algebra:

**Theorem 8.** *$P(B_1(H))_j$ is a Girard quantale with no unit.*

It might be thought that some completion precess allows to add units to Frobenius quantales. This is actually true, yet the resulting embedding does not preserve the negations. There is indeed a fundamental obstruction towards adding units:

**Theorem 9.** *Let $Q$ be a Frobenius quantale for which there exists a quantale embedding into a unital Frobenius quantale which also preserves negations. Then $\bigwedge_{x \in Q} x\backslash x$ is a unit of $Q$.*

In order to further understand the structure of unitless Frobenius quantales, we have investigated tight endomaps of $M_n$, the finite modular lattice with $n$ atoms which are also coatoms. We give characterizations of these endomaps and enumerate them. For a tight sup-preserving endomap $f$ of $M_n$, the implications $f\backslash f$ (one implication computed in the quantale of tight endomaps and the other computed in the quantale of all sup-preserving endomaps) coincide. This ensures reasonable properties of elements of the form $f\backslash f$, for example they are idempotent. It is easily argued, then, that elements of this form are not closed under infima. We do not know yet whether similar phenomena hold for quantales of tight endomaps of $L$ when $L$ is an arbitrary complete lattice.

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# Semilinear idempotent distributive $\ell$-monoids

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A *distributive $\ell$-monoid* is an algebra $\mathbf{M} = \langle M, \cdot, \wedge, \vee, e\rangle$ such that $\langle M, \cdot, e\rangle$ is a monoid, $\langle M, \wedge, \vee\rangle$ is a distributive lattice, and for all $a, b, c, d \in M$

$$a(b \wedge c)d = abd \wedge acd \quad \text{and} \quad a(b \vee c)d = abd \vee acd.$$

The class $\mathcal{DLM}$ of distributive $\ell$-monoids forms a variety (equational class). We call a distributive $\ell$-monoid *idempotent* or *commutative* if its monoid reduct is idempotent or commutative, respectively. A distributive $\ell$-monoid is called *semilinear* if it is isomorphic to a subdirect product of totally ordered monoids, i.e., distributive $\ell$-monoids where the lattice reduct is a total order. For a totally ordered monoid $\mathbf{M}$, we write $\mathbf{M} = \langle M, \cdot, e, \leq\rangle$, where $\leq$ is the lattice-order of $\mathbf{M}$. The class $\mathcal{SemDLM}$ of semilinear distributive $\ell$-monoids forms a variety that is generated by the class of totally ordered monoids and every subdirectly irreducible member of this variety is totally ordered. Moreover it is shown in [3] (see also [1]) that every commutative distributive $\ell$-monoid is semilinear.

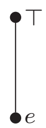
The aim of this work is to study the variety $\mathcal{SemIdDLM}$ of semilinear idempotent distributive $\ell$-monoids and its subvariety $\mathcal{CIdDLM}$ of commutative idempotent distributive $\ell$-monoids. Bearing in mind that $\mathcal{SemIdDLM}$ is locally finite, we use the $e$-sum construction of [4,5] (see also [2]) to investigate the structure of finite totally ordered idempotent monoids. Let $\mathbf{L} = \langle L, \cdot_L, e, \leq_L\rangle$ and $\mathbf{M} = \langle M, \cdot_M, e, \leq_M\rangle$ be totally ordered idempotent monoids, where we relabel the elements of $M$ and $L$ such that $M \cap L = \{e\}$. Then the *$e$-sum* of $\mathbf{L}$ and $\mathbf{M}$ is defined as $\mathbf{L} \oplus \mathbf{M} = \langle M \cup L, \cdot, e, \leq\rangle$, where $\cdot$ is the extension of the monoid operations $\cdot_L$ and $\cdot_M$ with $a \cdot b = b \cdot a = a$ for all $a \in L\setminus\{e\}$ and $b \in M$ and $\leq$ is the least extension of the orders $\leq_L$ and $\leq_M$ to $L \cup M$ that satisfies for all $a \in L\setminus\{e\}$, $b \in M$ that $a \leq b$ if $a \leq_L e$ and $b \leq a$ if $e \leq_L a$. The $e$-sum of two totally ordered idempotent monoids is again a totally ordered idempotent monoid and the operation of taking $e$-sums is associative. Accordingly we write $\bigoplus_{i=1}^{n} \mathbf{M}_i$ for $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_n$, where $\bigoplus_{i=1}^{0} \mathbf{M}_i \coloneqq \mathbf{0}$ is a trivial algebra. It turns out that every finite totally ordered idempotent monoid can be constructed as an $e$-sum using only the four algebras $\mathbf{C}_2$, $\mathbf{C}_2^{\partial}$, $\mathbf{G}_3$, and $\mathbf{D}_3$ described below.

$\mathbf{C}_2 = \langle\{\bot, e\}, \cdot, e, \leq\rangle$

| $\cdot$ | $e$ | $\bot$ |
|---|---|---|
| $e$ | $e$ | $\bot$ |
| $\bot$ | $\bot$ | $\bot$ |


$\mathbf{C}_2^{\partial} = \langle\{e, \top\}, \cdot, e, \leq\rangle$

| $\cdot$ | $e$ | $\top$ |
|---|---|---|
| $e$ | $e$ | $\top$ |
| $\top$ | $\top$ | $\top$ |




$\mathbf{G}_3 = \langle\{\bot, e, \top\}, \cdot, e, \leq\rangle$

| $\cdot$ | $e$ | $\bot$ | $\top$ |
|---|---|---|---|
| $e$ | $e$ | $\bot$ | $\top$ |
| $\bot$ | $\bot$ | $\bot$ | $\bot$ |
| $\top$ | $\top$ | $\top$ | $\top$ |


$\mathbf{D}_3 = \langle\{\bot, e, \top\}, \cdot, e, \leq\rangle$

| $\cdot$ | $e$ | $\bot$ | $\top$ |
|---|---|---|---|
| $e$ | $e$ | $\bot$ | $\top$ |
| $\bot$ | $\bot$ | $\bot$ | $\top$ |
| $\top$ | $\top$ | $\bot$ | $\top$ |

**Theorem.** Every finite totally ordered idempotent monoid is isomorphic to an $e$-sum $\bigoplus_{i=1}^{n} \mathbf{M}_i$ with $\mathbf{M}_i \in \{\mathbf{C}_2, \mathbf{C}_2^\partial, \mathbf{G}_3, \mathbf{D}_3\}$. Moreover, this $e$-sum is unique with respect to the algebras $\mathbf{C}_2$, $\mathbf{C}_2^\partial$, $\mathbf{G}_3$, $\mathbf{D}_3$.

We also characterize the finite subdirectly irreducibles of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$ in terms of $e$-sums.

**Theorem.** A finite totally ordered idempotent $\ell$-monoid $\mathbf{M}$ is subdirectly irreducible if and only if there exists an $n > 0$ and algebras $\mathbf{M}_i \in \{\mathbf{C}_2, \mathbf{C}_2^\partial, \mathbf{G}_3, \mathbf{D}_3\}$ for $i \in \{1, \dots, n\}$ such that $\mathbf{M} \cong \bigoplus_{i=1}^{n} \mathbf{M}_i$ and $\mathbf{M}_i = \mathbf{M}_{i+1}$ implies $\mathbf{M}_i \in \{\mathbf{G}_3, \mathbf{D}_3\}$ for every $i \in \{1, \dots, n-1\}$.

Using this characterization and [5, Corollary 4.3] we prove:

**Theorem.** The subvariety lattice of $\mathcal{S}em\mathcal{I}d\mathcal{DLM}$ is countably infinite.

For the commutative case the characterization of the finite subdirectly irreducibles yields that for every $n > 1$ the variety $\mathcal{CI}d\mathcal{DLM}$ contains up to isomorphism exactly two $n$-element subdirectly irreducibles which we denote by $\mathbf{C}_n$ and $\mathbf{C}_n^\partial$. Setting $\mathbf{C}_1 = \mathbf{C}_1^\partial$ to be a trivial algebra, we can give an explicit characterization of the subvariety lattice of $\mathcal{CI}d\mathcal{DLM}$.

**Theorem.** The subvariety lattice of $\mathcal{CI}d\mathcal{DLM}$ is of the following form, where $V(\mathbf{A})$ denotes the variety generated by $\mathbf{A}$:

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{CI}d\mathcal{DLM} \arrow[d, dotted, no head] & \\
 & V(\mathbf{C}_{n+1}) \vee V(\mathbf{C}_{n+1}^\partial) \arrow[dl, no head] \arrow[dr, no head] & \\
V(\mathbf{C}_{n+1}) \arrow[dr, no head] & & V(\mathbf{C}_{n+1}^\partial) \arrow[dl, no head] \\
 & V(\mathbf{C}_n) \vee V(\mathbf{C}_n^\partial) \arrow[d, dotted, no head] & \\
 & V(\mathbf{C}_3) \vee V(\mathbf{C}_3^\partial) \arrow[dl, no head] \arrow[dr, no head] & \\
V(\mathbf{C}_3) \arrow[dr, no head] & & V(\mathbf{C}_3^\partial) \arrow[dl, no head] \\
 & V(\mathbf{C}_2) \vee V(\mathbf{C}_2^\partial) \arrow[dl, no head] \arrow[dr, no head] & \\
V(\mathbf{C}_2) \arrow[dr, no head] & & V(\mathbf{C}_2^\partial) \arrow[dl, no head] \\
 & V(\mathbf{C}_1) &
\end{tikzcd}
$$

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# One-Sorted Program Algebras

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A *Kleene algebra* [3] is a structure $(K, \cdot, +, ^*, 1, 0)$ where $(K, +, \cdot, 1, 0)$ is an idempotent semiring and $^* : K \to K$, the Kleene star operation, satisfies

$$1 + x + x^*x^* \leq x^* \tag{1}$$

$$yx \leq x \implies y^*x \leq x \tag{2}$$

$$xy \leq x \implies xy^* \leq x \tag{3}$$

A Kleene algebra is $^*$-*continuous* iff

$$xy^*z = \sum_{n\geq 0} xy^n z\,. \tag{4}$$

Kleene algebra formalizes equational reasoning about regular languages and algebras of binary relations. Kleene algebras with tests [4], KAT, a two-sorted generalization of Kleene algebra containing a Boolean subalgebra of tests, formalizes equational reasoning about while programs.

*Kleene algebra with (Boolean) domain* KAD [1, 2] provides a one-sorted alternative to KAT. KAD expands KA with two unary operators $\mathsf{d}$ and $\mathsf{a}$ such that

$$x \leq \mathsf{d}(x)x \tag{5}$$

$$\mathsf{d}(xy) = \mathsf{d}(x\mathsf{d}(y)) \tag{6}$$

$$\mathsf{d}(x) \leq 1 \tag{7}$$

$$\mathsf{d}(0) = 0 \tag{8}$$

$$\mathsf{d}(x + y) = \mathsf{d}(x) + \mathsf{d}(y) \tag{9}$$

$$\mathsf{a}(x) + \mathsf{d}(x) = 1 \tag{10}$$

$$\mathsf{d}(x)\mathsf{a}(x) = 0 \tag{11}$$

A symmetric variant of KAD is *Kleene algebra with (Boolean) codomain*, KAC; its axiomatization results from the axiomatization of KAD by replacing $\mathsf{d}(x)x$ with $x\mathsf{c}(x)$ in the first axiom, $x\mathsf{d}(y)$ with $\mathsf{c}(x)y$ in the second axiom and $\mathsf{d}$ with $\mathsf{c}$ in the rest. In each Kleene algebra with domain, the test algebra $(\mathsf{d}(K), \cdot, +, \mathsf{a}, 1, 0)$ is a Boolean algebra, and similarly for $\mathsf{c}(K)$ in Kleene algebras with codomain. Consequently, the equational theory of KAT embeds to the equational theory of KAD (and KAC). However, from the viewpoint of Kleene algebra with tests, KAD and KAC have some peculiar features: the test algebra is necessarily the largest Boolean subalgebra of the negative cone (of the underlying Kleene algebra), and not every Kleene algebra expands to a Kleene algebra with domain (or codomain), the culprit being the locality axiom (6).

In the first part of the talk we introduce a generalization of KAD and KAC that avoids their peculiar features while retaining their good properties. *One-sorted Kleene algebra with tests* OneKAT expands Kleene algebra with two unary operations $\mathsf{t}$ and $\mathsf{t}'$ such that

$$\mathsf{t}(0) = 0 \tag{12}$$

$$\mathsf{t}(1) = 1 \tag{13}$$

$$\mathsf{t}(\mathsf{t}(x) + \mathsf{t}(y)) = \mathsf{t}(x) + \mathsf{t}(y) \tag{14}$$

$$\mathsf{t}(\mathsf{t}(x)\mathsf{t}(y)) = \mathsf{t}(x)\,\mathsf{t}(y) \tag{15}$$

$$\mathsf{t}(x)\mathsf{t}(x) = \mathsf{t}(x) \tag{16}$$

$$\mathsf{t}(x) \leq 1 \tag{17}$$

$$1 \leq \mathsf{t}'(\mathsf{t}(x)) + \mathsf{t}(x) \tag{18}$$

$$\mathsf{t}'(\mathsf{t}(x))\,\mathsf{t}(x) \leq 0 \tag{19}$$

$$\mathsf{t}'(\mathsf{t}(x)) = \mathsf{t}(\mathsf{t}'(\mathsf{t}(x))) \tag{20}$$

*Speaker.

We will show that the test algebra $(\mathsf{t}(K), \cdot, +, \mathsf{t}', 1, 0)$ of each $\mathsf{OneKAT}$ algebra is a Boolean algebra and that the equational theory of $\mathsf{KAT}$ embeds into the equational theory of $\mathsf{OneKAT}$; moreover, every Kleene algebra expands to a $\mathsf{OneKAT}$ algebra and the test algebra of a $\mathsf{OneKAT}$ algebra is not necessarily the maximal Boolean subalgebra of the negative cone of the underlying Kleene algebra. We will also show that adding "back" some $\mathsf{KAD}$ axioms—such as additivity (9), left preserver (5) or sublocality $\mathsf{t}(xy) \leq \mathsf{t}(x\mathsf{t}(y))$—does not change this.

In the second part of the talk, we consider a particular extension of $\mathsf{OneKAT}$ called $\mathsf{S}$-type $\mathsf{OneKAT}$ algebras, $\mathsf{SKAT}$. An $\mathsf{S}$-type $\mathsf{OneKAT}$ algebra is $(K, \cdot, +, \rightarrow, \hookrightarrow, {}^*, 1, 0, \mathsf{t}, \mathsf{e})$ where $(K, \cdot, +, \rightarrow, \hookrightarrow, {}^*, 1, 0)$ is a residuated Kleene algebra, that is

$$y \leq x \hookrightarrow z \iff xy \leq z \iff x \leq y \rightarrow z\,, \tag{21}$$

and $\mathsf{t}$, $\mathsf{e}$ are unary operators satisfying the following:

| | | | |
|---|---|---|---|
| $\mathsf{t}(\mathsf{t}(x)\mathsf{t}(y)) = \mathsf{t}(x)\,\mathsf{t}(y)$ | (15) | $\mathsf{t}(\mathsf{e}(x)) \leq x$ | (25) |
| $\mathsf{t}(x) \leq 1$ | (17) | $x \leq x\mathsf{t}(x)$ | (26) |
| $\mathsf{t}(x+y) = \mathsf{t}(x) + \mathsf{t}(y)$ | (22) | $\mathsf{t}(xy) \leq \mathsf{t}(\mathsf{t}(x)y)$ | (27) |
| $\mathsf{e}(x) \leq \mathsf{e}(x+y)$ | (23) | $\mathsf{t}(x \rightarrow y) \leq x \rightarrow x\mathsf{t}(y)$ | (28) |
| $x \leq \mathsf{e}(\mathsf{t}(x))$ | (24) | $1 \leq \mathsf{t}(\mathsf{t}(x) \rightarrow 0) + \mathsf{t}(x)$ | (29) |

We define $\mathsf{t}'(x) = \mathsf{t}(\mathsf{t}(x) \rightarrow 0)$. The operators $\mathsf{t}$ and $\mathsf{e}$ form a Galois connection. $\mathsf{SKAT}$ is an expansion of $\mathsf{KAC}$ with residuals and $\mathsf{e}$, and of Pratt's action algebras [6] with $\mathsf{t}$ and $\mathsf{e}$. $\mathsf{SKAT}$ is a variety since both (2–3) and (21) can be replaced by equations already in action algebras. The equational replacement of (2–3) is made possible by the presence of the residuals $\rightarrow$ and $\hookrightarrow$, together with the equational axioms of "pure induction", $(x \rightarrow x)^* \leq x \rightarrow x$ and $(x \hookrightarrow x)^* \leq x \hookrightarrow x$; see [6].

We show that the three-sorted substructural logic of partial correctness $\mathsf{S}$, introduced by Kozen and Tiuryn in [5], embeds into the equational theory of $^*$-continuous $\mathsf{SKAT}$ algebras.

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# Sums of Kripke frames and locally finite modal logics

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In classical model theory, there is a number of results ("composition theorems") that reduce the theory (first-order, MSO) of a compound structure (e.g., sum or product) to the theories of its components, see, e.g., [Gur85]. In this talk we discuss the composition method in the context of modal logic.

We consider the operation of sum on Kripke frames, where a family of frames-summands is indexed by elements of another frame. In many cases, the modal logic of sums inherits the finite model property and decidability from the modal logic of summands [BR10], [Sha18]. Under a general condition, the satisfiability problem on sums is polynomial space Turing reducible to the satisfiability problem on summands; in particular, for many modal logics decidability in PSpace is an immediate corollary from the semantic characterization of the logic [Sha22].

In this talk we announce the following result: if both the logic of indices and the logic of summands are locally finite, then the logic of sums is also locally finite. We also formulate a sufficient syntactic condition for local finiteness of bimodal logics.

## Main result

Fix an $\mathrm{A} < \omega$ for the alphabet of modal operators.

**Definition 1.** Consider a family $(\mathsf{F}_i)_{i\in I}$ of A-frames $\mathsf{F}_i = (W_i, (R_{i,a})_{a\in \mathrm{A}})$. The *sum* $\sum_{i\in \mathsf{I}} \mathsf{F}_i$ *of the family* $(\mathsf{F}_i)_{i\in I}$ *of* A-*frames over an* A-*frame* $\mathsf{I} = (I, (S_a)_{a\in\mathrm{A}})$ is the A-frame $(\bigsqcup_{i\in I} W_i, (R_a^{\Sigma})_{a\in\mathrm{A}})$, where $\bigsqcup_{i\in I} W_i = \bigcup_{i\in I}(\{i\}\times W_i)$ is the disjoint union of sets $W_i$, and

$$(i,w)R_a^{\Sigma}(j,v) \quad \text{iff} \quad (i=j \,\&\, wR_{i,a}v) \text{ or } (i\neq j \,\&\, iS_aj).$$

For classes $\mathcal{I}$, $\mathcal{F}$ of A-frames, let $\sum_{\mathcal{I}}\mathcal{F}$ be the class of all sums $\sum_{i\in\mathsf{I}}\mathsf{F}_i$ such that $\mathsf{I}\in\mathcal{I}$ and $\mathsf{F}_i\in\mathcal{F}$ for every $i$ in $\mathsf{I}$.

Modal logics of sums appear in various contexts such as provability logic, complexity and decision problems, completeness problems; see, e.g., [Bek10, Sha08, Bal09, BR10, Sha18, Sha22].

**Theorem 1.** *Let $\mathcal{F}$ and $\mathcal{I}$ be classes of* A-*frames. If the modal logics* $\mathrm{Log}(\mathcal{F})$ *and* $\mathrm{Log}(\mathcal{I})$ *are locally finite, then the logic* $\mathrm{Log}(\sum_{\mathcal{I}}\mathcal{F})$ *is locally finite as well.*

The proof is based on the semantic criterion of local finiteness given in [SS16] (Theorem 4.3).

## Lexicographic sums

The sum operation given above does not change the signature. In many cases it is convenient to characterize a polymodal logic via the following variant of the sum operation.

**Definition 2.** Let $\mathsf{I} = (I, S)$ be a unimodal frame, $(\mathsf{F}_i)_{i\in I}$ a family of A-frames, $\mathsf{F}_i = (W_i, (R_{i,a})_{a\in\mathrm{A}})$. The *lexicographic sum* $\overset{\mathrm{lex}}{\sum_{\mathsf{I}}}\mathsf{F}_i$ is the $(1+\mathrm{A})$-frame $\left(\bigsqcup_{i\in I} W_i, S^{\mathrm{lex}}, (R_a)_{a<N}\right)$, where

$$\begin{aligned}(i,w)S^{\mathrm{lex}}(j,u) \quad &\text{iff} \quad iSj,\\ (i,w)R_a(j,u) \quad &\text{iff} \quad i=j \;\&\; wR_{i,a}u.\end{aligned}$$

For a class $\mathcal{F}$ of A-frames and a class $\mathcal{I}$ of 1-frames, $\sum^{\text{lex}}_{\mathcal{I}}\mathcal{F}$ denotes the class of all sums $\sum^{\text{lex}}_{\mathsf{I}}\mathsf{F}_i$, where $\mathsf{I} \in \mathcal{I}$ and all $\mathsf{F}_i$ are in $\mathcal{F}$. For a unimodal $L_1$, let $\sum^{\text{lex}}_{L_1}L_2$ be the logic of the class $\sum^{\text{lex}}_{\text{Frames}\,L_1}\text{Frames}\,L_2$.

In the case when all summands are equal, this operation is the *lexicographic product*; lexicographic products of modal logics were introduced in [Bal09].

**Theorem 2.** *Let $L_1$ be a unimodal logic, $L_2$ be an* A*-modal logic. If $L_1$ and $L_2$ are locally finite, then the logic $\sum^{\text{lex}}_{L_1}L_2$ is locally finite as well.*

This theorem is an easy corollary of Theorem 1.

Consider the 2-modal formulas $\alpha = \Diamond_1\Diamond_0 p \to \Diamond_0 p$, $\beta = \Diamond_0\Diamond_1 p \to \Diamond_0 p$, $\gamma = \Diamond_0 p \to \Box_1\Diamond_0 p$. One can see that these formulas are valid in every lexicographic sum $\sum^{\text{lex}}_{\mathsf{I}}\mathsf{F}_i$ of 1-frames $\mathsf{F}_i$. In many cases, $\alpha, \beta, \gamma$ provide a complete axiomatization of $\sum^{\text{lex}}_{L_1}L_2$, that is we have

$$\sum^{\text{lex}}_{L_1}L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\}, \tag{1}$$

where $L_1 * L_2$ denotes the *fusion* of unimodal $L_1$ and $L_2$, $L + \Psi$ denotes the smallest normal logic containing $L \cup \Psi$. In particular, (1) holds for the logic $\sum^{\text{lex}}_{\text{GL}}\text{GL}$ [Bek10] (where GL is the Gödel-Löb logic) and for $\sum^{\text{lex}}_{\text{S4}}\text{S4}$ [Bal09].

**Theorem 3.** *Let $L_1$ and $L_2$ be locally finite canonical unimodal logics. If the class* Frames $L_1$ *is definable in first-order language without equality, then the logic $L_1 * L_2 + \{\alpha, \beta, \gamma\}$ is locally finite.*

The proof follows from the fact that under the condition of the theorem, (1) holds for $L_1$ and $L_2$.

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# Initiating descent theory for closure spaces

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By a closure space we mean a set equipped with a set of its subsets closed under arbitrary intersections. We describe various classes of maps of closure spaces that naturally occur in their descent theory, including a complete characterization of effective descent morphisms of finite closure spaces, which is our main result. It turns out that the analogy with such descriptions in the case of preorders [2] is stronger than in the intermediate case of topological spaces (see [1] and [7]). Nevertheless various general (category-theoretic and) topological tools used in [4], [10], [5], [8], and [9] turn out to be helpful also in the context of closure spaces. We also briefly consider connections with what we called strict monadic topology in [3].

Note that several different notions of a closure space have been introduced by various authors, some of them a long time ago; and it is interesting that the recent paper [6] almost enters descent theory: among many other things it describes pullback stable regular epimorphisms of closure spaces, slightly different ones, but that result can be copied in our context.

In finitely complete categories with coequalizers of equivalence relations we isolate the conditions that make a descent morphism to fail to be effective and translate them in terms of closure spaces. Then, the main result is based on the following observation specific to closure spaces, which has no counterpart neither for general topological spaces nor for preorders:

Let $E$ and $E'$ be closure spaces with the same underlying set such that the identity map $1_E$ is a continuous map from $E'$ to $E$, or, equivalently,

$$\overline{S}^{E'} \subseteq \overline{S}^{E} \text{ for all } S \subseteq E;$$

let $p : E \to B$ also be a continuous map of closure spaces. Then the following conditions are equivalent:

(a) the triple $(E', 1_E, \pi_1)$, where $\pi_1 : E \times_B E' \to E'$ is defined as the first projection, that is, by $\pi_1(e, e') = e$, is a descent data for $p$;

(b) $\overline{S}^{E} \cap p^{-1}p(\overline{p^{-1}p(S)}^{E'})) \subseteq \overline{S}^{E'}$ for all $S \subseteq E$.

Suppose the equivalent conditions (a) and (b) above are satisfied and let us write $p'$ for $p$ considered as a morphism from $E'$ to $B$. If both $p$ and $p'$ are regular epimorphisms, then, for every set $X$ closed as a subset of $E'$ but not as a subset of $E$, there exists a set $Y$ with the same property and $X \subset Y$ (strict inclusion).

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# Are finite affine topological spaces worthy of study?

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**Abstract**

Motivated by the theorem of S. A. Morris stating that every topological space is homeomorphic to a subspace of a product of a *finite* (3-element) topological space, the talk shows that this result is no longer valid in case of affine topological spaces (inspired by the concept of *affine set* of Y. Diers), which include, e.g., many-valued topology. In particular, we present an affine analogue of the original 3-element space and show its relationship to an affine analogue of the well-known Sierpinski space, both of which are (in general) infinite. Our message is that finite spaces play a (probably) less important role in affine topological setting (e.g., in many-valued topology) than they do in the classical topology.

## 1 Finite topological spaces

There exists a well-known result of S. A. Morris [10], stating that every topological space is homeomorphic to a subspace of a product of copies of the *Davey topological space* (or just *Davey space* in this talk) in the terminology of [10], which is a space $\mathcal{D} = (D, \tau_D)$ having a 3-element underlying set $D = \{0, 1, 2\}$ equipped with a topology $\tau_D = \{\emptyset, \{1\}, \{0, 1, 2\}\}$. Stating differently, $\mathcal{D}$ is an extremal coseparator in the category **Top** of topological spaces and continuous maps [1]. In view of this result and to answer the criticism of some researchers claiming that "finite topological spaces are not in the slightest bit interesting", it is stated in [10] that "perhaps there is something of interest in finite spaces after all".

There is another point supporting a general interest in finite topological spaces, i.e., the concept of *Sierpinski space* $\mathcal{S} = (\{0, 1\}, \{\emptyset, \{1\}, \{0, 1\}\})$. This concept plays an important role in general topology. One of its simple but crucial properties is the fact that a topological space is $T_0$ if and only if it can be embedded into some power of $\mathcal{S}$. Stating differently, $\mathcal{S}$ is an $\mathcal{M}$-coseparator in the category $\mathbf{Top}_0$ of $T_0$ topological spaces, where $\mathcal{M}$ stands for the class of all topological embeddings (i.e., initial injective maps) in $\mathbf{Top}_0$. Motivated by the importance of the notion of Sierpinski space, E. G. Manes introduced its analogue for concrete categories under the name of *Sierpinski object* [9]. Restated in the language of [1], an object $S$ of a concrete category $\mathbf{C}$ is said to be a *Sierpinski object* provided that for every $\mathbf{C}$-object $C$, the hom-set $\mathbf{C}(C, S)$ is an initial source.

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As a recent development, [6, 7] showed that finite topological spaces are precisely the *finite preorders* (i.e., finite sets equipped with a reflexive and transitive binary relation). It appeared that nearly all the results of topological descent theory could be motivated by their finite instances, which became simple and natural when expressed in the language of finite preorders.

## 2 Affine topological spaces

Induced by a considerable number of different approaches to *lattice-valued* (or *many-valued*) topology and a clear lack of intercommunication means between them, an affine approach to lattice-valued topology has been introduced in [11], taking its origin in the notion of *affine set* of Y. Diers [4]. More precisely, while a classical topological space $(X, \tau)$ consists of a set $X$ equipped with a topology $\tau$, which, being a subset of the powerset $\mathcal{P}X$ of $X$, has the algebraic structure of *frame* [8], the affine approach replaces the standard contravariant powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{CBAlg}^{op}$ from the category **Set** of sets to the dual of the category of complete Boolean algebras with a functor $T : \mathbf{X} \to \mathbf{A}^{op}$ from a category $\mathbf{X}$ to the dual category of a variety of algebras $\mathbf{A}$, and requires $\tau$ to be a subalgebra of $TX$. Taking a suitable variety $\mathbf{A}$ and an appropriate functor $T$, one obtains not only the classical topological spaces, but also, e.g., the closure spaces of [2] as well as the most essential lattice-valued topological frameworks.

This talk tries to investigate the role of *finite* spaces in affine topology. More precisely, there already exists an affine analogue of the Sierpinski space in terms of the Sierpinski object of E. G. Manes [3, 5, 11], which (in general) is no longer finite. This talk provides an affine analogue of the Davey space and shows its simple relation to the affine Sierpinski space. Since the affine Davey space is (in general) no longer finite as well, the simple message we want to convey here is that finite spaces play a (probably) less important role in affine topological setting (for example, in lattice-valued topology) than they do in the classical topology.

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# Kan-injectivity and KZ-doctrines

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In a category of algebras for a given finitary signature, the reflection of an algebra $A$ into a subcategory presented by a set of finitary equations (or implications) may be constructed by means of a transfinite chain of morphisms which converges at step $\omega$. This construction may be performed in any locally presentable category, leading to a solution of the Orthogonal Subcategory Problem for a set of morphisms, as it was proved by Gabriel and Ulmer [9]. Thus, given a set $\mathcal{H}$ of morphisms, we conclude that the full subcategory of orthogonal objects constitutes the category of Eilenberg-Moore algebras of an idempotent monad. This result can be extended to a broader context, including locally bounded categories, as shown in [11].

In [6], Banaschewski and Herrlich observed that, for an algebra, the satisfaction of an implication is equivalent to the injectivity of the algebra with respect to a certain morphism. The combination of this idea with the properties of the above transfinite chain gave rise to the study of deduction systems where the "formulas" are morphisms, see [1], [2], [3].

In a 2-category $\mathcal{K}$, an object $X$ is said to be *left Kan-injective* with respect to a morphism $h : A \to B$ if, for every morphism $f : A \to X$, there is a left Kan extension of $f$ along $h$ given by an invertible 2-cell:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\cong}{\Longrightarrow}"] & B \arrow[dl, "\mathrm{Lan}_h(f)"] \\ X & \end{tikzcd}$$

A morphism $u : X \to Y$ is said to be *left Kan-injective* with respect to $h : A \to B$ if it preserves left Kan extensions of morphisms $f : A \to X$ along $h$, that is, $\mathrm{Lan}_h(uf) = u\mathrm{Lan}_h(f)$. We denote by $\mathrm{LInj}(\mathcal{H})$ the locally full subcategory of $\mathcal{K}$ of all objects and all morphisms left Kan injective with respect to every morphism of $\mathcal{H}$.

In 2-categories, certain conditions on objects, and on morphisms, may be given by means of left Kan injectivity. For instance, in the 2-category **Pos**, made of posets, monotone maps and pointwise order between maps, the posets with binary suprema and the morphisms which preserve them are precisely those which are left Kan-injective with respect to the embedding of the discrete poset $D$ with two elements into the poset obtained from $D$ by joining a top element. Taken in the 2-category **Cat** of categories, this embedding presents, via left-Kan-injectivity, the categories with binary coproducts and morphisms preserving them. Many other examples may be find in [5], [7], [8], [10] and [13].

Starting from a set $\mathcal{H}$ of morphisms in a 2-category $\mathcal{K}$, we construct, for each object $X$, a transfinite chain leading to the components of the unit of a lax-idempotent pseudo-monad (or KZ-doctrine, see [11] and [12]). The algebras and the homomorphisms of this KZ-doctrine are,

*Speaker.

essentially, the objects and the morphisms of $\mathcal{K}$ which are left Kan-injective with respect to $\mathcal{H}$. This encompasses, as a particular case, the transfinite chains mentioned above, and generalizes the Kan-injective reflection chain presented in [5] for order-enriched categories, which, in [4], was a key tool for obtaining a Kan-injectivity logic.

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# The coordinatization of the spectra of $\ell$-groups

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Let $G$ be a lattice ordered Abelian group (henceforth, just $\ell$-group), the set of prime ideals (=prime convex $\ell$-subgroups) of $G$, endowed with the hull-kernel topology is called the *spectrum* of $G$ and is denoted $\mathtt{Spec}\, G$. Spectra of $\ell$-groups have received much attention in the past. It is know that they are *generalised spectral spaces* —i.e. $T_0$, sober, and with a basis of compact open sets— with the additional property of being *completely normal* —i.e., if $x$ and $y$ are in the closure of a point $z$, then either $x$ is in the closure of $y$ or $y$ is in the closure of $x$.

Recently in [4] it was proved that the above properties characterise the second-countable spectra of $\ell$-groups; in the same paper it was proved that there cannot be any first order axiomatisation of the distributive lattices that are dual to spectra of $\ell$-groups.

It was also observed that spectra of $\ell$-groups do not retain enough information to characterise the $\ell$-group they come form; e.g., $\mathbb{Z}$ and $\mathbb{R}$ have the same spectrum (a single point). Here we propose a way to attach further information on $\mathtt{Spec}$ so to be able to reconstruct the original $\ell$-group, up to isomorphism. More specifically, we provide a way to *coordinatize* $\mathtt{Spec}$. Indeed, although $\mathtt{Spec}$ cannot always be embedded in some power $\mathbb{R}^k$, because of its topological structure, we show that a coordinatization is possible if one enlarges the set of coordinates to an *ultrapower* of $\mathbb{R}$. This is due to the following result.

**Theorem 1.** *For every cardinal $\alpha$ there exists an ultrapower of $\mathbb{R}$ on an $\alpha$-regular ultrafilter, in which all linearly ordered groups of cardinality smaller than $\alpha$ embed.*

Since quotients over prime ideals are linearly ordered, one obtains the wanted embedding.

Let $\mathcal{U}$ be an arbitrary ultrapower of $\mathbb{R}$. Any power $\mathcal{U}^k$ can be endowed with a Zariski-like topology by taking as a basis of closed sets:

$$\mathbb{V}(t(x)) \coloneqq \{u \in \mathcal{U}^k \mid \mathcal{U} \models t(u) = 0\},$$

with $t(x)$ ranging among $k$-ary terms in the language of $\ell$-groups. It is easy to see that this topology is not $T_0$, however it is sober and has a basis of compact open sets. Moreover it is compact only because the origin $O$ belongs to all closed sets. For any ultrapower $\mathcal{U}$ let $\overline{\mathcal{U}^k}$ be the largest $T_0$ quotient of $\mathcal{U}^k \setminus \{O\}$.

**Theorem 2.** *For any $k$, the space $\overline{\mathcal{U}^k}$ is a* generalized spectral space*. Moreover for any $\ell$-group $G$, there exist a cardinal $k$ and an ultrapower $\mathcal{U}$ of $\mathbb{R}$, such that $\mathtt{Spec}\, G$ is homeomorphic to a closed subspace of $\overline{\mathcal{U}^k}$.*

This result has two consequences. The first is that $\mathtt{Spec}$ induces a duality on $\ell$-groups — more details about this are contained in another abstract submitted by the same authors. The second consequence is a characterisation of spectra of $\ell$-groups as closed subspaces of some $\overline{\mathcal{U}^k}$.

**Theorem 3.** *Let $X$ be any generalised spectral space. The space $X$ is the spectrum of some $\ell$-group if and only if $X$ is, up to an iso, a closed subspace of $\overline{\mathcal{U}^k}$ for some ultrapower $\mathcal{U}$ of $\mathbb{R}$ and some cardinal $k$.*

*Speaker.

A useful tool to understand the geometry of $\mathcal{U}^n$, with $n \in \mathbb{N}$, is provided by the following decomposition theorem.

**Theorem 4** ([2]). *If $a \in \mathcal{U}^n$, then $a = \alpha_1 v_1 + \ldots + \alpha_t v_t$ where $\alpha_1, \ldots, \alpha_t \in \mathcal{U}$ are positive, $\alpha_{i+1}/\alpha_i$ is infinitesimal, $v_1, \ldots, v_t$ are orthonormal vectors in $\mathbb{R}^n$, and the decomposition is unique.*

An important contribution to the study of the space of prime ideals of an $\ell$-group is [3] (see also [1] for the case with strong unit). There, prime ideals of finitely generated free $\ell$-groups are characterised as the sets of piecewise (homogeneous) linear functions with integer coefficients that vanish on a *cone* determined by a tuple of vectors. Using an adaptation of Theorem 4 we are able to connect our results with the above description and provide a more intuitive version using concepts of non-standard analysis. Indeed, we are able to associate to any non-standard point a tuple of orthonormal vectors in $\mathbb{R}^n$, called *index*. This induces a correspondence between prime ideals and sets of indexes. For every prime ideal $\mathfrak{p}$, the set associated to $\mathfrak{p}$ in this way turns out to be the set of all indexes of all non-standard points that are images of $\mathfrak{p}$ under all possible embeddings in Theorem 2, or equivalently, the closure of any image of $\mathfrak{p}$ under these embeddings.

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# Associativity in Quantum Logic

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Recent years have seen the emergence of a research program aimed at applying the tools and techniques of substructural logic and residuated structures to longstanding problems in quantum logic (see, e.g., [2, 3, 1]). In this context, [4] introduced residuated ortholattices as an abstract algebraic environment for studying residuation in quantum logic. Formally, a *residuated ortholattice* is an algebra $(A, \wedge, \vee, \backslash, \neg, 0, 1)$ such that $(A, \wedge, \vee, \neg, 0, 1)$ is an ortholattice and $\backslash$ is a one-sided residual of the (non-commutative and non-associative) Sasaki product operation given by $x \cdot y = x \wedge (\neg x \vee y)$, i.e., for all $x, y, z \in A$,

$$x \wedge (\neg x \vee y) = x \cdot y \leq z \iff y \leq x \backslash z.$$

In any residuated ortholattice $\mathbf{A}$, one may define a negation operation $\sim x = x\backslash 0$ that satisfies the De Morgan laws. It is shown in [4] that the image of any residuated ortholattice $\mathbf{A}$ under the associated closure operator $x \mapsto \sim\sim x$ is an orthomodular lattice, called the *orthomodular skeleton* of $\mathbf{A}$, and moreover that every orthomodular lattice arises in this fashion. Further, this fact supports a double negation translation interpreting orthomodular lattices in residuated ortholattices, similar to the well known negative translation of classical logic into intuitionistic logic. Accordingly, a better understanding of the structure of residuated ortholattices promises to deepen our understanding of quantum logic, and particularly the relationship between quantum and substructural logics.

Here we offer a preliminary study of the structure of residuated ortholattices, focusing on residuated ortholattices whose orthomodular skeletons are Boolean as a natural entry point. We generalize the classical theory of the commutating elements in orthomodular lattices to residuated ortholattices. Using these technical results, we obtain the following.

**Theorem 1.** *Sasaki product is associative in a residuated ortholattice $\mathbf{A}$ if and only if the orthomodular skeleton of $\mathbf{A}$ is a Boolean algebra.*

This theorem further highlights the importance of weak associative properties in the study of residuated ortholattices, which was already implicit in [4]. Commensurately, we study various kinds of weak associativity in residuated ortholattices and provide a host of sufficient conditions on $x, y, z$ that guarantee $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ in arbitrary residuated ortholattices. In particular, we obtain the following.

**Theorem 2.** *Let $\mathbf{A}$ be a residuated ortholattice and let $a, b, c \in A$. If any two of $a, b, c$ have the same image under the map $x \mapsto \sim\sim x$, then $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.*

Weak associative properties have been studied extensively in the context of orthomodular lattices [7, 6, 5], where many results on weak forms of associativity are obtained by checking that they hold in free algebras. Our results apply a fortiori in the the orthomodular case.

*Speaker.

In addition to generalizing most of the known results on weak associativity in orthomodular lattices, we obtain simple equational proofs of these theorems for the first time.

It is well known that Sasaki product is associative in an orthomodular lattice **A** if and only if **A** is Boolean, as is reflected in Theorem 1 above. This poses the natural question of what equations hold in the $\cdot$ reducts of residuated ortholattices in the presence of associativity. By deploying the structural results obtained so far, we obtain the following.

**Theorem 3.** *Let $\mathcal{A}$ be the subvariety of residuated ortholattices with associative Sasaki product, and let $\varepsilon$ be an equation containing only variables and the operation $\cdot$ of Sasaki product. Then $\varepsilon$ holds in $\mathcal{A}$ if and only if $\varepsilon$ holds in all left-regular bands.*

In other words, the variety generated by the $\cdot$ reducts of members of $\mathcal{A}$ is precisely the variety of left-regular bands. This observation opens up the possibility of applying the techniques developed in the study of left-regular bands to residuated ortholattices and quantum logic.

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# Canonical extensions and the frame of fitted sublocales

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## 1 Sublocales of the frame of strongly exact filters

When we see a frame $L$ as a pointfree space its *sublocales* are its pointfree subspaces. Sublocales are certain subsets of $L$, and these form a coframe $\mathsf{S}(L)$ when ordered under set inclusion. We have open sublocales, closed sublocales, and fitted sublocales; they correspond, respectively, to open, closed, and saturated subsets of a space. For a frame $L$ we have an order anti-isomorphism between the collection of fitted sublocales $\mathsf{Fitt}(L)$ and the frame $\mathsf{SEFilt}(L)$ of the *strongly exact* filters (see [3]).

I will show in this talk that the frame of strongly exact filters has some distinguished sublocales, which correspond to special subcolocales of $\mathsf{Fitt}(L)$ because of the result above. We have a diagram of sublocale inclusions as follows.

$$\begin{tikzcd}
\mathsf{MCFilt}(L) \arrow[r, "\subseteq"] & \mathsf{EFilt}(L) \arrow[dr, "\subseteq"] & & \\
& & \mathsf{SEFilt}(L) \arrow[r, "\subseteq"] & \mathsf{Filt}(L) \\
\mathsf{MCPFilt}(L) \arrow[r, "\subseteq"] & \mathsf{MSOFilt}(L) \arrow[ur, "\subseteq"] & &
\end{tikzcd}$$

Here $\mathsf{MCFilt}(L)$ is the frame of meets of *closed* filters, that is, filters of the form $\{x \in L : x \vee a = 1\}$ for some $a \in L$. The frame $\mathsf{EFilt}(L)$ is the frame of *exact* filters, a notion that appears in [3] and is in a certain sense dual to that of strongly exact filter. The frame $\mathsf{MCPFilt}(L)$ is the frame of meets of completely prime filters, and the frame $\mathsf{MSOFilt}(L)$ is the frame of meets of Scott-open filters.

The diagram above corresponds to the following diagram of subcolocale inclusions.

$$\begin{tikzcd}
\mathit{fit}[S_{\mathfrak{c}}(L)] \arrow[r, "\subseteq"] & \mathit{fit}[\mathsf{S}_b(L)] \arrow[dr, "\subseteq"] & \\
& & \mathsf{Fitt}(L) \\
\mathit{fit}[\mathsf{sp}[\mathsf{S}(L)]] \arrow[r, "\subseteq"] & \mathit{fit}[\mathsf{S}_k(L)] \arrow[ur, "\subseteq"] &
\end{tikzcd}$$

*Speaker.

Here $\mathsf{S_c}(L)$ is the collection of joins of closed sublocales (see [4]); $\mathsf{S}_b(L)$ is the coframe of joins of complemented sublocales (whose structure has been recently studied in [5]). $\mathsf{S}_k(L)$ is the collection of joins of compact sublocales, and $\mathsf{sp}[\mathsf{S}(L)]$ is the coframe of spatial sublocales. Finally, *fitt* : $\mathsf{S}(L) \to \mathsf{S}(L)$ denotes the *fitting* operator (see [2]), which is the closure coming from $\mathsf{Fitt}(L) \subseteq \mathsf{S}(L)$ seen as a closure system.

# 2 The connection with canonical extensions

Canonical extensions for frames have been recently studied in [1]. The canonical extension of a frame is got as a *polarity*. A *polarity* is a complete lattice $\mathsf{Pol}(X, Y, R)$ such that $X$ and $Y$ are sets and $R \subseteq X \times Y$ is a relation, and such that we have two canonical maps $f_X : X \to \mathsf{Pol}(X, Y, R)$ and $f_Y : Y \to \mathsf{Pol}(X, Y, R)$ satisfying a certain universal property. The sublocales of $\mathsf{Fitt}(L)$ above are all instances of polarities. In general, in fact, we have that for a collection $\mathcal{S} \subseteq \mathsf{S}(L)$ of sublocales and for the collection $\mathsf{Op}(L)$ of open sublocales

$$\mathsf{Pol}(\mathcal{S}, \mathsf{Op}(L), \subseteq) = \mathit{fitt}[\mathfrak{J}(\mathcal{S})],$$

where $\mathfrak{J}$ denotes closure under all joins. The theory of polarities also helps us see a symmetry between fitted and closed sublocales. We have in general that

$$\mathsf{Pol}(\mathcal{S}, \mathsf{Cl}(L), \subseteq) = cl[\mathfrak{J}(\mathcal{S})].$$

This means that the structures above all enjoy a universal property, all variations of the defining universal property of the canonical extension of a frame. This leads to the following result exhibiting a symmetry between the closures *fit* and $cl$, which were recently compared in [2]. Here for a frame $L$ the expression $\mathfrak{B}(L)$ denotes the Booleanization of $L$. For a frame $L$ we have:

1. $\mathsf{Pol}(\mathsf{Cl}(L), \mathsf{Op}(L), \subseteq) \cong \mathit{fit}[\mathsf{S}_c(L)] = \mathfrak{B}(\mathsf{Fitt}(L))$;
2. $\mathsf{Pol}(\mathsf{Op}(L), \mathsf{Cl}(L), \subseteq) \cong cl[\mathsf{Op}(L)] = \mathfrak{B}(\mathsf{Cl}(L)) \cong \mathfrak{B}(L)$.

Since this is work in progress, in the end of the talk I will outline some open questions, especially related to how much more interaction we can find between the theory of polarities and the closure systems within $\mathsf{S}(L)$.

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# The Dependence Problem in Varieties of Modal Semilattices

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In [1] De Jongh and Chagrova introduced the notion of dependence for intuitionistic propositional logic IPC. Formulas $\varphi_1, \ldots, \varphi_n$ are called IPC-*dependent* if there exists a formula $\psi(p_1, \ldots, p_n)$ such that $\vdash_{\mathsf{IPC}} \psi(\varphi_1, \ldots, \varphi_n)$ and $\nvdash_{\mathsf{IPC}} \psi(p_1, \ldots, p_n)$, otherwise $\varphi_1, \ldots, \varphi_n$ are called IPC-*independent*.

In [4] we generalise this notion to a universal algebra setting. Let $\mathcal{L}$ be an algebraic language and let $\mathcal{V}$ be a variety of $\mathcal{L}$-algebras. By $\mathrm{Tm}(\bar{x})$ and $\mathrm{Eq}(\bar{x})$, we denote the set of $\mathcal{L}$-terms and $\mathcal{L}$-equations over the set of variables $\bar{x}$, respectively. We call terms $t_1, \ldots, t_n \in \mathrm{Tm}(\bar{x})$ $\mathcal{V}$-*dependent* if for some equation $\varepsilon(y_1, \ldots, y_n)$,

$$\mathcal{V} \vDash \varepsilon(t_1, \ldots, t_n) \quad \text{and} \quad \mathcal{V} \nvDash \varepsilon;$$

otherwise, we call $t_1, \ldots, t_n$ $\mathcal{V}$-*independent*. This notion of dependence is related to a general algebraic notion of dependence introduced by Marcewski in [3]. The problem of deciding whether any finite number of terms are $\mathcal{V}$-dependent is called the *dependence-problem for* $\mathcal{V}$.

Let $\Gamma, \Delta \subseteq \mathrm{Eq}(\bar{y})$. We write $\Gamma \mathrel{|\!\!\sim}_{\mathcal{V}} \Delta$ if for any substitution $\sigma\colon \mathrm{Tm}(\bar{y}) \to \mathrm{Tm}(\omega)$ extended to equations by setting $\sigma(s \approx t) = \sigma(s) \approx \sigma(t)$,

$$\mathcal{V} \vDash \sigma(\Gamma) \quad \Longrightarrow \quad \mathcal{V} \vDash \sigma(\delta) \text{ for some } \delta \in \Delta.$$

Then we say that a set $\Delta \subseteq \mathrm{Eq}(\bar{y})$ is $\mathcal{V}$-*refuting* for $\bar{y}$ if for any equation $\varepsilon(\bar{y})$,

$$\mathcal{V} \nvDash \varepsilon \quad \Longleftrightarrow \quad \{\varepsilon\} \mathrel{|\!\!\sim}_{\mathcal{V}} \Delta.$$

**Lemma 1.** *For any $\mathcal{V}$-refuting set $\Delta(\bar{y})$ for $\bar{y} = \{y_1, \ldots, y_n\}$, the terms $t_1, \ldots, t_n \in Tm(\bar{x})$ are $\mathcal{V}$-dependent if and only if $\mathcal{V} \vDash \delta(t_1, \ldots, t_n)$ for some $\delta \in \Delta$.*

Thus, for varieties that have a decidable equational theory and for which a finite $\mathcal{V}$-refuting set for any finite $\bar{y}$ can be constructed, the dependence-problem is decidable.

**Example 2.** We define $[n] \coloneqq \{1, \ldots, n\}$. Let us consider the variety $\mathcal{L}at$ of all lattices and let

$$\Delta_n \coloneqq \Big\{ y_i \le \bigvee_{j \in [n] \setminus \{i\}} y_j \mid i \in [n] \Big\} \cup \Big\{ \bigwedge_{j \in [n] \setminus \{i\}} y_j \le y_i \mid i \in [n] \Big\}.$$

We can show that $\Delta_n$ is a $\mathcal{V}$-refuting set for $\{y_1, \ldots, y_n\}$ and thus, the dependence-problem for $\mathcal{L}at$ is decidable.

Let us now turn our attention to modal semilattices studied for example in [2]. First we consider $\mathcal{MJS}$, the variety of $\langle A, \vee, \Box \rangle$-algebras such that $\langle A, \vee \rangle$ is a semilattice and

$$\Box a \vee \Box b \le \Box(a \vee b), \quad \text{for all } a, b \in A.$$

The approach used for $\mathcal{L}at$ does not quite work for $\mathcal{MJS}$, since the modal depth of a formula can be arbitrarily large, but a more general approach works.

**Lemma 3.** *The following set of $\mathcal{MJS}$-inequations in $\bar{y}$ is $\mathcal{MJS}$-refuting for $\bar{y}$:*

$$\Delta_{\bar{y}} \coloneqq \{y \leq s \mid y \in \bar{y} \text{ and } s \neq s_1 \vee y \vee s_2\} \cup \{\Box^k y \leq y' \mid y, y' \in \bar{y} \text{ and } k \geq 0\}.$$

Note that $\Delta_{\bar{y}}$ is an infinite set of non-valid inequations for any $\bar{y} \neq \varnothing$. Let $\mathrm{md}(t)$ denote the modal depth of the term $t$ and define $\mathrm{md}(s \leq t) = \max\{\mathrm{md}(s), \mathrm{md}(t)\}$ for the inequation $s \leq t$.

**Theorem 4.** *Let $t_1, \dots, t_n \in Tm(\bar{x})$ and let $\bar{y} = \{y_1, \dots, y_n\}$. Then $t_1, \dots, t_n$ are $\mathcal{MJS}$-dependent if and only if there is an inequation $\delta \in \Delta^d_{\bar{y}}$ such that*

$$\mathcal{MJS} \vDash \delta(t_1, \dots, t_n),$$

*where $\Delta^d_{\bar{y}} \coloneqq \{\delta \in \Delta_{\bar{y}} \mid md(\delta) \leq d\}$ and $d \coloneqq \max\{md(t_1), \dots, md(t_n)\}$.*

**Corollary 5.** *The dependence-problem for $\mathcal{MJS}$ is decidable.*

Furthermore, we consider $\mathcal{MMS}$, the variety of $\langle A, \wedge, \Box\rangle$-algebras such that $\langle A, \wedge\rangle$ is a semilattice and

$$\Box a \wedge \Box b = \Box(a \wedge b), \quad \text{for all } a, b \in A.$$

Again, the approach used for $\mathcal{L}at$ does not work for $\mathcal{MMS}$, but studying free $\mathcal{MMS}$-algebras yields a simple method for deciding the dependence-problem for $\mathcal{MMS}$. The free $\mathcal{MMS}$-algebra over $m > 0$ generators is isomorphic to the following $\mathcal{MMS}$-algebra:

$$\langle (\mathcal{P}_{fin}(\mathbb{N}))^m \backslash \{\langle \varnothing, \dots, \varnothing\rangle\}, \cup, \Box\rangle,$$

where $\mathcal{P}_{fin}(\mathbb{N})$ is the set of all finite subsets of $\mathbb{N}$, and $\cup, \Box$ are defined component-wise with $\Box\{a_1, \dots, a_k\} \coloneqq \{a_1+1, \dots, a_k+1\}$ for $a_1, \dots, a_k \in \mathbb{N}$. Let $\mathbf{F}(\bar{x})$ denote the free $\mathcal{MMS}$-algebra over $\bar{x}$ and let the elements of $\mathbf{F}(\bar{x})$ be of the form $[t]$ for terms $t \in \mathrm{Tm}(\bar{x})$.

**Theorem 6.** *Let us consider the $\mathcal{MMS}$-terms $t_1, \dots, t_n \in Tm(\bar{x})$. The following are equivalent:*

1. *$t_1, \dots, t_n$ are $\mathcal{MMS}$-dependent.*
2. *There is $i \in \{1, \dots, n\}$ such that for each variable $x$ occurring in $t_i$ one of the following holds:*
   (a) *$[t_i] = [\Box^k x \wedge \Box^l x \wedge t'_i] \in \mathbf{F}(\bar{x})$, where $k \neq l$.*
   (b) *There is $j \in \{1, \dots, n\} \backslash \{i\}$ such that $x$ also occurs in $t_j$.*

**Corollary 7.** *The dependence-problem for $\mathcal{MMS}$ is decidable.*

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# Abstracting sheafification as a tripos-to-topos adjunction

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In topos theory, localic toposes constitute an important class of examples of Grothendieck's toposes of sheaves. A localic topos is defined as the category $\mathsf{Shv}(A)$ of shaves on a locale $A$. Recall that sheaves are usually described as *presheaves on $A$ satisfying a glueing condition*, and it is well-known that the category of sheaves $\mathsf{Shv}(A)$ and that of presheaves $\mathsf{PShv}(A)$ are connected via the so-called *sheafification* functor, i.e. the left adjoint functor $s$ to the inclusion

$$\begin{tikzcd} \mathsf{PShv}(A) \arrow[r, bend left, "s"] \arrow[r, phantom, "\perp"] & \mathsf{Shv}(A). \arrow[l, bend left, "i"] \end{tikzcd}$$

Motivated by the construction of the category of sheaves on a locale, Hyland, Johnstone and Pitts introduced in [2] a generalization of this construction called *tripos-to-topos* construction. The idea is that given a tripos $P\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{Pos}$, acronym of *topos-representing indexed pre-orders sets*, we can construct a topos $\mathsf{T}_P$ that generalises the notion of topos of sheaves on a locale. This is a proper generalization since each tripos-to-topos construction produces a Lawvere-Tierney elementary topos, but not necessarily a localic topos or more generally a Grothendieck topos. A remarkable example of such non-sheaf toposes are realizability toposes.

The main purpose of our work is to show that the sheafification adjunction happens to be the instance of a more abstract adjunction between tripos-to-topos constructions induced from an adjunction between two triposes in the sense of [2] satisfying suitable conditions. In detail:

**Theorem.** *Let $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ be a tripos such that*

1. *$\mathcal{C}$ has weak dependent products;*
2. *the predicate classifier $\Omega$ has a power object $\mathrm{P}\Omega$ in $\mathcal{C}$;*
3. *$\mathcal{C}$ admits a proper factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$, such that every epi of $\mathcal{E}$ splits.*

*We associate to $P$ a tripos $P^{\exists}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ and its tripos-to-topos construction $\mathsf{T}_{P^{\exists}}$, called* ***presheaf topos substitute****, for which*

1. *the tripos-to-topos construction $\mathsf{T}_P$ is included in $\mathsf{T}_{P^{\exists}}$ via $i\colon \mathsf{T}_P \longrightarrow \mathsf{T}_{P^{\exists}}$;*
2. *the inclusion $i\colon \mathsf{T}_P \longrightarrow \mathsf{T}_{P^{\exists}}$ has a left adjoint*

$$\begin{tikzcd} \mathsf{T}_{P^{\exists}} \arrow[r, bend left, "s"] \arrow[r, phantom, "\perp"] & \mathsf{T}_P. \arrow[l, bend left, "i"] \end{tikzcd}$$

*Speaker.

As corollaries, we obtain the following (where $\mathsf{Set}$ is the category of sets and functions definable in the classical set theory ZFC):

**Corollary.** *For every tripos* $P\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *on* $\mathsf{Set}$ *we have an adjunction of toposes*

$$\begin{tikzcd} \mathsf{T}_{P^{\exists}} \arrow[r, bend left, "s"] \arrow[r, phantom, "\perp"] & \mathsf{T}_{P}. \arrow[l, bend left, "i"] \end{tikzcd}$$

**Corollary.** *Let* $A^{(-)}\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ *be the localic tripos. Then the category* $\mathsf{T}_{A^{\exists}}$ *is precisely the topos of presheaves* $\mathsf{PShv}(A)$*, and the arrow* $s\colon \mathsf{PShv}(A) \longrightarrow \mathsf{Shv}(A)$ *is precisely the sheafification.*

The key tools to prove our results is the characterization of the full existential completion and its tripos-to-topos presented in [5], the notion of quotient completion in [4] and adiunctions between them in [3]: given a tripos $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$, the doctrine $P^{\exists}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$ is defined as the full existential completion of the tripos $P$ . Under the assumptions of our previous theorem, we show that $P_{\exists}$ is a tripos (and then $\mathsf{T}_{P^{\exists}}$ is a topos). The adjunction between $\mathsf{T}_{P^{\exists}}$ and $\mathsf{T}_{P}$ follows from the universal properties of the full existential completion and elementary quotient completions [3]. From a result presented in [5], since $P^{\exists}$ is the full existential completion of $P$, we conclude that $\mathsf{T}_{P^{\exists}} \equiv (\mathcal{G}_P)_{\mathsf{ex/lex}}$ where $\mathcal{G}_P$ is the Grothendieck category of $P$.

In the localic case, this result together with the well-known equivalence $\mathsf{PShv}(A) \equiv (A_+)_{\mathsf{ex/lex}}$, see [1, 6], and the observation that $A_+ \equiv \mathcal{G}_A$ where $\mathcal{G}_A$ is the Grothendieck category of the localic tripos $A^{(-)}\colon \mathsf{Set}^{\mathrm{op}} \longrightarrow \mathsf{InfSl}$, see [5], allows us to conclude that $\mathsf{T}_{A^{\exists}} \equiv \mathsf{PShv}(A)$ and that the adjunction we obtain is precisely the sheafification one.

A remarkable non-localic example of doctrinal sheaves is the effective topos $\mathsf{Eff} \equiv \mathsf{T}_R$, where $R$ is the realizability tripos, whose doctrinal presheaves is $\mathsf{T}_{R^{\exists}}$. We leave to future work to investigate whether $\mathsf{T}_{R^{\exists}}$ is itself a realizability topos, too.

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# A Gödel-type translation for non-distributive logics

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By "non-distributive logics" (aka LE-logics), we understand all logics the algebraics semantics of which is given by varieties of normal lattices-expansions. In [4], a type of (Kripke-style) relational semantics for LE-logics was discussed, which is based on reflexive directed graphs (i.e. tuples $(Z, E)$ such that $Z$ is a set and $E \subseteq Z \times Z$ is reflexive). In the same paper, it was suggested that graph-based semantics supports a conceptual interpretation of LE-logics as *hyperconstructive* logics of evidential reasoning. The proposed talk, which is part of a research program on extending results from intuitionistic logics to LE-logics, reports on a work in progress aimed at extending the Gödel-McKinsey-Tarski (GMT) translation and related results to LE-logics.

As discussed in [5], the semantic underpinning of the GMT translation [6, 9] of intuitionistic logic into the classical normal modal logic S4 is the observation that partial orders $\mathbb{F} = (W, \leq)$ serve as "Kripke frames" for both logics. The difference lies in how the *complex algebra* is defined in each case: when $\mathbb{F}$ is understood as an intuitionistic frame, then the complex algebra of $\mathbb{F}$ is defined as $\mathcal{P}^{\uparrow}(W)$, i.e. the perfect Heyting algebra of the upward-closed subsets of $W$; when $\mathbb{F}$ is understood as an S4-frame, then the complex algebra of $\mathbb{F}$ is defined as $(\mathcal{P}(W), \Diamond_{\leq})$, i.e. the perfect BAO of the subsets of $W$ with $\Diamond_{\leq}$ satisfying the axioms corresponding to reflexivity and transitivity.

On the algebraic side, for any S4 modal algebra $\mu = (B, \Box)$, the corresponding Heyting algebra is defined by $\mathbb{H}_{\mu} = (H_{\mu}, \wedge, \vee, \rightarrow)$, where $H_{\mu} = \{\Box a, a \in B\}$ and $a \rightarrow b = \Box(\neg a \vee b)$. On the other hand, given a Heyting algebra $\mathbb{H} = (H, \wedge_H, \vee_H, \rightarrow_H)$, its corresponding S4 modal algebra is given by $\mu_{\mathbb{H}} = (B(\mathbb{H}), \Box)$, where $B(\mathbb{H})$ is the free Boolean extension of $\mathbb{H}$ and for any $\alpha = \bigwedge_{i=1}^{m}(\neg c_i \vee d_i)$ for some $c_i, d_i \in H$, we set $\Box(\alpha) = \bigwedge_{i=1}^{m}(c_i \rightarrow_H d_i)$. The maps $\phi : \mu \rightarrow \mathbb{H}_{\mu}$, and $\psi : \mathbb{H} \rightarrow \mu_{\mathbb{H}}$, provide a starting point for comparing the varieties of S4 modal algebra and Heyting algebras, which has given rise to several transfer theorems and the Blok-Esakia theorem [3, 2, 8, 10, 1].

Recently, in [7], a GMT-type translation between sorted modal logic and lattice logic was explored based on the polarity-based semantics for LE-logics. In our presentation, reflexive graphs $\mathbb{X} = (Z, E)$ serve as relational frames both for propositional lattice logic and for classical normal modal logic T. The difference again lies in how the *complex algebra* is defined: when $\mathbb{X}$ is understood as a lattice logic frame (i.e. a graph-based frame), then the complex algebra $\mathbb{A}$ of $\mathbb{X}$ is defined as the concept lattice of the polarity $\mathbb{P}_{\mathbb{X}} = (Z_A, Z_X, I_{E^c})$; when understood as an S4-frame, the complex algebra $\mathbb{B}$ of $\mathbb{X}$ is defined as $(\mathcal{P}(W), \Diamond_E, \blacklozenge_E)$, i.e. the perfect BAO of the subsets of $W$ with $\Diamond_E, \blacklozenge_E$ satisfying the axiom corresponding to reflexivity and the adjunction related properties between them. We define the GMT-translation $\tau = (\tau_1, \tau_2)$, where $\tau_1, \tau_2 : \mathcal{L}_{\mathrm{LL}} \rightarrow \mathcal{L}_{\mathrm{T}}$ by the following recursion:

$$
\begin{array}{rclrcl}
\tau_1(p) & = & \rhd\blacktriangleright p & \tau_2(p) & = & \neg\blacktriangleright p \\
\tau_1(\bot) & = & \rhd\blacktriangleright\bot & \tau_2(\bot) & = & \bot \\
\tau_1(\top) & = & \top & \tau_2(\top) & = & \neg\blacktriangleright\top \\
\tau_1(\phi\wedge\psi) & = & \tau_1(\phi)\wedge\tau_1(\psi) & \tau_2(\phi\vee\psi) & = & \tau_2(\phi)\vee\tau_2(\psi) \\
\tau_1(\phi\vee\psi) & = & \rhd(\blacktriangleright\tau_1(\phi)\wedge\blacktriangleright\tau_1(\psi)) & \tau_2(\phi\wedge\psi) & = & \neg\blacktriangleright(\rhd\neg\tau_2(\phi)\wedge\rhd\neg\tau_2(\psi)).
\end{array}
$$

Where $\blacktriangleright := \neg\blacklozenge_E$ and $\rhd := \neg\Diamond_E$. Unlike the intuitionistic case, two maps $\tau_1$, $\tau_2$, are needed to capture the satisfaction and co-satisfaction relation of LE-logics:

**Proposition 1.** *For every $\mathcal{L}_{\mathrm{LL}}$-formula $\phi$, and every reflexive graph $\mathbb{X} = (Z, E)$,*

$$\mathbb{X} \Vdash \phi \quad \textit{iff} \quad \mathbb{X} \Vdash^* \tau_1(\phi),$$
$$\mathbb{X} \succ \phi \quad \textit{iff} \quad \mathbb{X} \nVdash^* \tau_2(\phi).$$

Similary to the intuitionistic case that admissible valuations are restricted to upwards-closed sets, in the LE-logic case the admissible valuations are restricted to Galois-stable sets of the polarity $(Z_A, Z_X, I_{E^c})$.

On the algebraic side, for any reflexive tense modal algebra $\mathbb{A}$, we define the corresponding lattices $\rho_1(\mathbb{A}) = \mathbb{L}_1 = (\mathrm{L}_1, \vee_1, \wedge_1, \top_1, \bot_1)$, $\rho_2(\mathbb{A}) = \mathbb{L}_2 = (\mathrm{L}_2, \vee_2, \wedge_2, \top_2, \bot_2)$ as follows:

For any $a, b \in \mathbb{A}$,

$$\mathrm{L}_1 = \{\rhd\blacktriangleright a \mid a \in \mathbb{A}\},$$
$$\mathrm{L}_2 = \{\neg\blacktriangleright a \mid a \in \mathbb{A}\},$$

$$\begin{array}{rclrcl} \rhd\blacktriangleright a \vee_1 \rhd\blacktriangleright b & = & \rhd(\blacktriangleright\rhd\blacktriangleright a \wedge \blacktriangleright\rhd\blacktriangleright b) & \rhd\blacktriangleright a \wedge_1 \rhd\blacktriangleright b & = & \rhd\blacktriangleright a \wedge \rhd\blacktriangleright b, \\ \neg\blacktriangleright a \vee_2 \neg\blacktriangleright b & = & \neg\blacktriangleright a \vee \neg\blacktriangleright b & \neg\blacktriangleright a \wedge_2 \neg\blacktriangleright b & = & \neg\blacktriangleright(\rhd\blacktriangleright a \wedge \rhd\blacktriangleright b). \end{array}$$

**Proposition 2.** *For any reflexive tense modal algebra $\mathbb{A}$, $\rho_1(\mathbb{A}) \cong \rho_2(\mathbb{A})$.*

However, unlike intuitionistic logic, given a lattice $\mathbb{L}$, there does not exist a free reflexive tense modal algebra expanding $\mathbb{L}$. In this presentation, we will explain why a free object doesn't exist by defining two minimal distinct tense modal algebras given $\mathbb{L}$, we will discuss the similarities and differences with the intuitionistic case and the comparison of the lattices of corresponding varieties. Finally, we will report on ongoing work on the generalization of the Blok-Esakia theorem and different translation theorems for the GMT translation between the reflexive tense modal logic and the general lattice logic.

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# Graded modal logic with a single modality

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**Graded modal logic** is an extension of classical modal logic with graded modalities $\Diamond_n (n \in \mathbb{N}^+)$ that allows to count the number of successors of a given state in a Kripke model. Intuitively, the formula $\Diamond_n A$ is satisfied at a point $w$ of a Kripke frame if and only if $w$ has at least $n$ successors satisfying $A$.

Graded modal logic was originally introduced in Goble [10]. Kaplan [12] studied graded modal logic as an extension of **S5**. The completeness of graded modal logic and its extensions was investigated in [9, 7, 2]. Van der Hoek [15] and Cerrato [3] used filtrations to obtain the finite model property and decidability of graded modal logic. Van der Hoek [15] also studied the expressibility, definability and correspondence theory. Bisimulations for graded modal logic were introduced in [8], and used to provide an alternative proof of the finite model property, and show that a first-order formula is invariant under graded bisimulation iff it is equivalent to a graded modal formula. Aceto, Ingolfsdottir and Sack [1] showed that resource bisimulation and graded bisimulation coincide over image-finite Kripke frames. Finally, various notions of epistemic and dynamic graded modal logics have been investigated in [16] and [13].

Even though the modality $\Diamond_1$ corresponds to the standard classical modal logic connective, and therefore retains all its properties, the modalities $\Diamond_n$ for $n \geq 2$ do not. In particular, the modalities $\Diamond_n$ are monotone, i.e. they satisfy the rule $\vdash \varphi \to \psi / \vdash \Diamond_n \varphi \to \Diamond_n \psi$, and satisfy $\Diamond_n \bot \leftrightarrow \bot$, but are not **additive**, that is, the implication $\Diamond_n (p \vee q) \to (\Diamond_n p \vee \Diamond_n q)$ fails for $n \geq 2$. Modal logics with monotone modalities have been extensively studied [4, 11, 14]. However, not much work has been done regarding the connections between monotonic modal logics and graded modal logic. In [6], building on the proof-theoretic and algebraic analysis of non-normal modal logics of [5], a line of research studying these connection was initiated, where an elementary but not modally definable class of neighbourhood frames was shown to exactly correspond to graded Kripke frames, and the notion of graded bisimulation was recasted through the lens of neighbourhood bisimulations.

This presentation reports on work that adds to the study of connections between monotonic modal logic and graded modal logic, albeit towards a different direction. Specifically, the standard axiomatization of graded modal logic relies on the **interaction** of the different graded modalities, and captures the properties of addition of natural numbers. However, when viewed as monotone modalities, each graded modality can also be studied in isolation. Accordingly, for every $n \in \mathbb{N}^+$, we introduce the logic $\mathcal{L}_n$, whose language contains a single modal operation, $\Diamond$, and whose theory is defined as the set of validities on Kripke frames, where $\Diamond$ is interpreted as the graded modality $\Diamond_n$ described in the first paragraph. We show that they have the finite model property, are decidable, and are finitely axiomatizable. We also show that for $n \neq m$ the logics $\mathcal{L}_n$ and $\mathcal{L}_m$ are distinct and in particular:

**Theorem 1.** *Let $n < m$ such that $m - 1 = (n - 1) \cdot k + r$ where $r < n - 1$. Then, if $r < k$ it follows that $\mathcal{L}_m \subsetneq \mathcal{L}_n$. If $k \leq r$, then there exists $\zeta_n, \theta_n \in \Phi$, such that $\zeta_n \in \mathcal{L}_n$ while $\zeta_n \notin \mathcal{L}_m$ and $\theta_n \in \mathcal{L}_m$ while $\theta_n \notin \mathcal{L}_n$.*

We will also discuss complete axiomatizations for these logics. In particular, assuming that $\alpha_1$, $\alpha_2$, $\alpha_3$ are mutually contradictory and likewise for $\beta_1$ and $\beta_2$, and denoting

$$\Diamond_1^{\psi} \varphi := \Diamond(\varphi \vee \psi) \wedge \neg \Diamond \psi, \tag{1}$$

*Speaker.

the following theorems hold:

**Theorem 2.** *The system consisting of all propositional tautologies, the monotonicity rule*

$$(M) \quad \vdash (p \to q) / \vdash (\Diamond p \to \Diamond q),$$

*and the following axioms*

$$\begin{array}{ll} (\bot) & \Diamond\bot \to \bot, \\ (G2) & \left[\Diamond_1^{q_1}(\alpha_1) \wedge \Diamond_1^{q_2}(\alpha_2)\right] \to \Diamond(\alpha_1 \vee \alpha_2), \end{array}$$

*and closed under modus ponens and uniform substitution is a sound and complete axiomatization of $\mathcal{L}_2$.*

**Theorem 3.** *The system consisting of all propositional tautologies, the monotonicity rule*

$$(M) \quad \vdash (p \to q) / \vdash (\Diamond p \to \Diamond q),$$

*and the following axioms*

$$\begin{array}{ll} (\bot) & \Diamond\bot \to \bot, \\ (G3_1) & \left[\Diamond_1^{q_1}(\alpha_1) \wedge \Diamond_1^{q_2}(\alpha_2) \wedge \Diamond_1^{q_3}(\alpha_3)\right] \to \Diamond(\alpha_1 \vee \alpha_2 \vee \alpha_3), \\ (G3_2) & \left[\Diamond_1^{q_1}(\alpha_2) \wedge \Diamond_1^{q_2}(\beta_2) \wedge \Diamond(\alpha_1 \vee \beta_1) \wedge \neg\Diamond(\alpha_1 \vee \alpha_2)\right] \to \Diamond(\beta_1 \vee \beta_2), \end{array}$$

*and closed under modus ponens and uniform substitution is a sound and complete axiomatization of $\mathcal{L}_3$.*

Finally we will discuss possible techniques for obtaining axiomatizations for $\mathcal{L}_n$ for $n \geq 4$.

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# Probability via logic: semantic analysis and proof theory

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Providing good proof systems for probabilistic logics is a long standing problem in proof theory and logics for uncertainty. In 1990, [5] introduces a logic to reason about probabilities and its Hilbert style calculus that contains three types of axioms and rules: the ones that govern the arithmetical part, i.e., the reasoning about inequalities; the ones that axiomatise probabilities; and the rules and axioms of classical propositional logic. The proposed calculus has the advantage of being quite intuitive and easy to use, however, its axiomatisation is infinite. In 2020, [1] utilises a two-layered modal logic to formalise reasoning about probabilities. The proposed calculus consists of three parts: the rules and axioms of the logic of events (i.e. classical logic) or 'inner logic'; the 'outer logic' that formalises reasoning with probabilities; and finally, the modalities that transform events into probabilistic statements.

This work is a part of larger research project aimed at providing good proof systems for probabilistic logics and other logics of uncertainty in a uniform and modular way. In this project, we use a generalization of display calculi introduced by Belnap [2]. This choice is motivated by the following two reasons. Firstly, display calculi are by design modular, insofar they implement a neat division of labour between logical rules (introducing the connectives and relying on their minimal order-theoretic properties) and so-called structural rules (capturing the specific features of the logic under consideration). Secondly, they provides a framework in which cut-elimination, a crucial property of proof systems, can be proved in a principled way as an application of a general meta-theorem.

The main difficulties in applying the theory of display calculi to the probability logics lies in the handling of the operators like $+$ and $-$ (i.e. the (truncated) sum and difference, respectively) and their interaction with the probability operator $P$.

A specific and preliminary difficulty in the handling of probability operator is that it is a non-normal operator (with usual definition of join and meet on interval $[0, 1]$), so standard techniques and design choices are immediately banned. To overcome this issue, we use a generalization of the standard theory to a so-called multi-type environment (see, for instance, [7] as a prototypical examples of this approach). Here we consider an equivalent multi-type presentation of the algebraic semantics for probability logic, thanks to which we can define an appropriate formal translations of the original language into a new multi-type language (preserving validity and derivability of formulas).

A similar problem arises when considering the peculiar axiom of probability logic (involving the sum, the difference and the probability operator as well) given that it not **analytic inductive** [6] in the original language, so, once again, standard techniques cannot be applied.

In an ongoing work, we introduce a generalization of standard display calculi we used to capture Łucasewicz logic and, in particular, to deal with the peculiar axiom of the logic involving the sum (or, equivalently, the subtraction depending on the presentation of the axiom). This axiom is key to capture the interaction with the probability operator, and the specific design choices implemented in this calculus (motivated by an algebraic analysis of Łucasewicz logic) are imported here as well. Moreover, we use a specialized version of the algorithm ALBA to automatically generate the analytic structural rules equivalently capturing the axiom. Showing that such rule preserve the analyticity of the basic calculus is work in progress.

Below we expand on the treatment of the probability operator. The key idea is that the non-normal operators (like the conditional binary operator of conditional logics or the monotone unary modalities in non-normal modal logics) can be decomposed into the composition of normal modal operators [4]. In this work, we use a similar approach to deal with the probability operator $P$.

Let $\mathcal{B}$ be any set and $\mathcal{P}(\mathcal{B})$ be its power-set. Let $P : \mathcal{P}(\mathcal{B}) \to [0, 1]$ be a probability function on it.

$$
\begin{tikzcd}
& \mathcal{P}\mathcal{P}(\mathcal{B}) \arrow[dr, "{[\not\leq]}"] & \\
\mathcal{P}(\mathcal{B}) \arrow[ur, "\langle\notin\rangle"'] \arrow[rr] \arrow[dr, "{[\in]}"'] & & {[0,1]} \\
& \mathcal{P}\mathcal{P}(\mathcal{B}) \arrow[ur, "\langle\leq\rangle"'] &
\end{tikzcd}
$$

Let $R_\in, R_\notin \subseteq \mathcal{P}(\mathcal{B}) \times \mathcal{B}$ be defined as follows. For any $a \in \mathcal{B}$, $A \in \mathcal{P}(\mathcal{B})$,

$$AR_\in a \quad \text{iff} \quad a \in A \quad \text{and} \quad AR_\notin a \quad \text{iff} \quad a \notin A.$$

Let $R_\leq, R_{\not\leq} \subseteq [0,1] \times \mathcal{P}(\mathcal{B})$ be defined as follows. For any $\alpha \in [0,1]$, $A \in \mathcal{P}(\mathcal{B})$,

$$\alpha R_\leq A \quad \text{iff} \quad \alpha \leq P(A) \quad \text{and} \quad \alpha R_{\not\leq} A \quad \text{iff} \quad \alpha \not\leq P(A).$$

Let $A \subseteq \mathcal{B}$, and $U \subseteq \mathcal{P}(\mathcal{B})$ be any subsets of $\mathcal{B}$ and $\mathcal{P}(\mathcal{B})$ respectively. Let $[\in](A) = [R_\in](A)$, $\langle\notin\rangle(A) = \langle R_\notin\rangle(A)$, $\langle\leq\rangle(U) = \langle R_\leq\rangle(U)$, and $[\not\leq](U) = [R_{\not\leq}](U)$, where for any relation $R$, $[R]$ and $\langle R\rangle$ denote the box and diamond operators corresponding to the relation $R$ on the given frame. Then, we have

**Lemma 1.** *For any $A \subseteq \mathcal{B}$, and $U \subseteq \mathcal{P}(\mathcal{B})$ we have (1) $[\in](A) = A^\downarrow$, (2) $\langle\notin\rangle(A) = (A^\uparrow)^c$, (3) $\langle\leq\rangle(U) = [0, \max\{P(A) \mid A \in U\}]$, and (4) $[\not\leq](U) = [0, \min\{P(A) \mid A \in U^c\}]$.*

The following corollary follows immediately from the Lemma.

**Corollary 2.** *For any $A \subseteq \mathcal{B}$, $P(A) = \max(\langle\leq\rangle[\in](A)) = \max([\not\leq]\langle\notin\rangle(A))$.*

Thus, under the identification of an interval with its largest element above, the corollary shows that the probability operator $P$ can be decomposed into the combination of normal operators $\langle\leq\rangle$, $[\in]$, $[\not\leq]$, and $\langle\notin\rangle$ in two ways. This decomposition allows us to write the probability axioms in the language of Łucasewicz logic expanded with the above modal operators. Therefore, the axioms of probability logic can be expressed in the above multi-type normal modal logic. We believe these techniques would allow us to introduce display-like calculi for probability logics and other (non-classical) logics of uncertainty such as the logics for probabilities and belief functions over Belnap-Dunn logic introduced in [3].

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# Duality, unification, and admissibility in the positive fragment of Łukasiewicz logic

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MV-algebras, the equivalent algebraic semantics in the sense of Blok-Pigozzi of Łukasiewicz logic, have a deep connection with interesting geometrical objects. Indeed, the category of finitely presented MV-algebras with homomorphisms is dually equivalent to a category whose objects are rational polyhedra and the morphisms are so-called $\mathbb{Z}$-maps [12]. This connection allows the study of relevant algebraic and logical properties from the geometrical point of view, such as, for instance, the study of projective algebras and amalgamation, or correspondingly, the investigation of interpolation and unification problems [6, 8, 9, 12, 14].

Looking at Łukasiewicz logic as a substructural logic, thus as an axiomatic extension of the Full Lambek Calculus with exchange and weakening [10], we consider its positive (i.e., 0-free) fragment. The latter is also algebraizable, and its corresponding equivalent algebraic semantics is the variety of Wajsberg hoops. Wajsberg hoops are interesting structures also from a purely algebraic point of view. They play an important role in the theory of hoops [4], which are naturally ordered commutative monoids, and they have a particular connection with lattice-ordered abelian groups (abelian $\ell$-groups for short). In fact, the variety $\mathsf{WH}$ of Wajsberg hoops is generated by its totally ordered members, that are, in loose terms, either negative cones of abelian $\ell$-groups, or *intervals* of abelian $\ell$-groups [2] (equivalently, MV-algebras, via Mundici's $\Gamma$ functor [13]). In the context of the algebraic semantics of many-valued logics, the relevance of Wajsberg hoops is also related to the study of the equivalent algebraic semantics of Hájek Basic Logic and its positive subreducts (BL-algebras and basic hoops). Given the well-known decomposition result in terms of Wajsberg hoops for totally ordered BL-algebras given by Aglianò and Montagna [1], the understanding of Wajsberg hoops is key to obtain interesting results in this framework.

We show that finitely presented Wajsberg hoops have an interesting geometrical dual as well, in particular, with what we will call *pointed* rational polyhedra. More precisely, we show how finitely presented Wajsberg hoops are dually equivalent to a (non-full) subcategory of rational polyhedra with $\mathbb{Z}$-maps, given by rational polyhedra in unit cubes $[0,1]^n$ that contain the lattice point $\mathbf{1} = (1, \ldots, 1)$, and *pointed* $\mathbb{Z}$-maps, that are $\mathbb{Z}$-maps that respect the lattice point $\mathbf{1}$. In particular, we use a key result in [2] to first show that Wajsberg hoops are equivalent to a (non full) subcategory of finitely presented MV-algebras, and then we suitably restrict the Marra-Spada duality [12].

The connection with the MV-algebraic duality with rational polyhedra allows the use of the deep results obtained by Cabrer and Mundici about finitely generated projective MV-algebras [8, 9, 6] to describe finitely generated projective Wajsberg hoops. In particular, we show that no MV-algebra (or more precisely, its 0-free reduct) is projective in the variety of Wajsberg hoops, and actually that finitely generated nontrivial projective Wajsberg hoops are necessarily unbounded. Interestingly enough, this implies that, in particular, the (0-free reduct of the)

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two-element Boolean algebra **2** is not projective in the variety of residuated lattices, while **2** is projective in every variety of bounded commutative integral residuated lattices, and in the variety of all bounded commutative integral residuated lattices it is the only finite projective algebra [3].

The fact that Wajsberg hoops are the equivalent algebraic semantics of the positive fragment of Łukasiewicz logic allows us to use our algebraic and geometric investigation to derive some analogies and differences between Łukasiewicz logic and its positive fragment. In particular, while deducibility in the fragment coincides with deducibility of positive terms in Łukasiewicz logic, the same is not true for admissibility of rules. That is, there are rules involving positive terms that are admissible in the positive fragment but not in Łukasiewicz logic.

Moreover, via the algebraic approach to unification problems developed by Ghilardi [11], we will show that the unification type of the variety of Wajsberg hoops, and thus of the positive fragment of Łukasiewicz logic, is nullary. This is in close analogy with the case of MV-algebras, and indeed our proof adapts to pointed rational polyhedra the pathological example given in [12] for the case of Łukasiewicz logic. Furthermore, via the algebraic approach to admissibility developed in [7], we show that while the exact unification type of Łukasiewicz logic is finitary, the one of its positive fragment is unitary. This in particular implies decidability of the admissibility of rules in Wajsberg hoops and in the positive fragment of Łukasiewicz logic.

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# Subordination Algebras as Semantic Environment of Input/Output Logic

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Input/output logic [11] has been introduced as a formal framework for modelling the interaction between logical inferences and other agency-related notions such as conditional obligations, goals, ideals, preferences, actions, and beliefs. This framework has been applied mainly in the context of the formalization of normative systems in philosophical logic and AI. Although, initially, this framework was intended "not [for] studying some kind of non-classical logic, but [as] a way of using the classical one", its generality and versatility makes it very suitable to support a range of enhancements in its expressiveness, such as those brought about by the addition of modal operators. Moreover, recently, there has been an interest in studying the interaction between the agency-related notions mentioned above with various forms of *nonclassical* reasoning [13, 14]. This interest has contextually motivated the introduction of algebraic and proof-theoretic methods in the study of input/output logic [15].

In this talk, we contribute to the latter research direction in the mathematical background of input/output logic (as defined in [11]) by introducing an algebraic semantics for it, based on (generalizations of) *subordination algebras* [1]. These can be defined as tuples $(A, \prec)$ such that $A$ is a Boolean algebra and $\prec$ is a binary relation on $A$ such that the direct (resp. inverse) image of each element $a \in A$ is a filter (resp. an ideal) of $A$. Subordination algebras are equivalent presentations of pre-contact algebras [10] and quasi-modal algebras [2, 3]. Since their introduction, subordination algebras have been systematically connected with various modal algebras (i.e. Boolean algebras expanded with semantic modal operators). This has made it possible to endow various modal languages with algebraic semantics based on subordination algebras, and use these languages to axiomatize the properties of these subordination algebras. In particular, Sahlqvist-type canonicity for modal and tense formulas on subordination algebras has been studied in [8] using topological techniques; in [9], using algebraic techniques, the canonicity result of [8] was strengthened and captured within the more general notion of canonicity in the context of *slanted algebras*, which was established using the tools of *unified correspondence theory* [5, 6, 7]. Slanted algebras are based on general lattices, and encompass variations and generalizations of subordination algebras such as those very recently introduced by Celani in [4], which are based on distributive lattices, and for which Celani develops duality-theoretic and correspondence-theoretic results.

In this talk, we propose a semantic framework in which the subordination relations $\prec$ of (proto-)subordination algebras interpret the normative system $N$ of input/output logic. This makes it possible to conceptually interpret the meaning of $\prec$ in terms of the behaviour of systems of norms, to systematically relate rules of $N$ with properties of $\prec$, and to interpret the output operators induced by $N$ as the modal operators associated with $\prec$. We characterize a number of basic properties of $N$ (or of $\prec$) in terms of modal axioms in this language. Interestingly, some of these properties are well known in the literature of input/output logic, since they capture intuitive desiderata about the interaction between norms and logical reasoning; some other properties originate from purely mathematical considerations, and have been dually characterized in Celani's correspondence-theoretic results in [4]. Thanks to the embedding of subordination algebras in the more general environment of slanted algebras, we have a mathematical environment for systematically exploring the interaction between norms and different modes of reasoning, and a systematic connection with families of logical languages which can be applied in different contexts to axiomatize the behaviour of various generalized systems of norms; this more general environment allows to also encompass results such as Celani's correspondence for subordination lattices as consequences of *standard* Sahlqvist modal correspondence.

Similar to the duality between necessity and possibility in modal logic, the notion of conditional permission (sometimes referred to as negative permission) has been introduced as dual to conditional obligation: "a code permits something with respect to a condition iff it does not forbid it under that same condition, i.e. iff the code does not require the contrary under that condition" [12]. The duality between subordination relations and pre-contact relations [10] allows us to propose precontact relations for modelling conditional permission. Time permitting, we will discuss the relationship between pre-contact algebra and different concepts of permissions proposed by Makinson and van der Torre [12].

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# Epimorphisms in Varieties of De Morgan Monoids*

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Let $\mathsf{K}$ be a variety of algebras. A homomorphism $f : \mathbf{A} \to \mathbf{B}$, with $\mathbf{A}, \mathbf{B} \in \mathsf{K}$, is called a $\mathsf{K}$-*epimorphism* if, whenever $\mathbf{C} \in \mathsf{K}$ and $g$ and $h$ are homomorphisms from $\mathbf{B}$ to $\mathbf{C}$ such that $g \circ f = h \circ f$, then $g = h$. All surjective homomorphisms in $\mathsf{K}$ are $\mathsf{K}$-epimorphisms. We say $\mathsf{K}$ has the *epimorphism surjectivity (ES) property* if the converse holds as well.

When a variety $\mathsf{K}$ algebraizes a logic $\vdash$, then $\mathsf{K}$ has the ES property if and only if $\vdash$ has the *(infinite deductive) Beth (definability) property* [5], i.e., whenever a set of variables is defined *implicitly* in terms of other variables by means of some formulas over $\vdash$, then it can also be defined *explicitly*.

A *De Morgan monoid* $\mathbf{A} = \langle A; \cdot, \wedge, \vee, \neg, e\rangle$ comprises a distributive lattice $\langle A; \wedge, \vee\rangle$, a commutative monoid $\langle A; \cdot, e\rangle$ satisfying $x \leqslant x \cdot x$, and a function $\neg \colon A \to A$, called an *involution*, such that $\mathbf{A}$ satisfies $\neg\neg x = x$ and $x \cdot y \leqslant z \iff x \cdot \neg z \leqslant \neg y$. (The derived operation $x \to y := \neg(x \cdot \neg y)$ turns $\mathbf{A}$ into a residuated lattice in sense of [3].) The class $\mathsf{DMM}$ of all De Morgan monoids is a variety that algebraizes the (substructural) relevance logic $\mathbf{R^t}$ of Anderson and Belnap [1].

We shall provide an overview of what is known about the ES property in varieties of De Morgan monoids. Some results are published in [8], and some will be presented in a forthcoming paper. Famously, Urquhart showed that $\mathsf{DMM}$ does *not* have the ES property [10]. We shall provide two sets of sufficient conditions for subvarieties that *do*. These results therefore settle natural questions about Beth-style definability for a range of substructural logics, in particular, for extensions of the relevance logic $\mathbf{R}^t$.

The first set of sufficient conditions will now be explicated. Let $\mathbf{A}$ be a De Morgan monoid. We say $\mathbf{A}$ is *negatively generated* if it is generated (as an algebra) by its set $\{a \in A \colon a \leqslant e\}$ of *negative* elements.

Recall that $F \subseteq A$ is a *deductive filter* of $\mathbf{A}$ if it is a lattice filter containing $e$. In this case we say $F$ is *prime* when $F = A$ or $A \setminus F$ is a lattice ideal. Let $\mathrm{Pr}(\mathbf{A})$ be the set of prime deductive filters $F$ of $\mathbf{A}$. We define the *depth* of $F$ in $\mathrm{Pr}(\mathbf{A})$ to be the greatest $n \in \omega$ (if it exists) such that there is a chain in $\mathrm{Pr}(\mathbf{A})$ of the form $F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = A$. If no such $n$ exists, we say $F$ has *depth* $\infty$ in $\mathrm{Pr}(\mathbf{A})$. We define $\mathrm{d}(\mathbf{A}) = \sup\{\mathrm{d}(F) : F \in \mathrm{Pr}(\mathbf{A})\}$. If $\mathsf{K}$ is a variety of De Morgan monoids, we define $\mathrm{d}(\mathsf{K}) = \sup\{\mathrm{d}(\mathbf{A}) : \mathbf{A} \in \mathsf{K}\}$. If a variety of De Morgan monoids is finitely generated, then it has finite depth (but not conversely).

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**Theorem 1** ([8])**.** *Let* $\mathsf{K}$ *be a variety of De Morgan monoids of finite depth such that each finitely subdirectly irreducible member of* $\mathsf{K}$ *is negatively generated. Then* $\mathsf{K}$ *has the ES property.*

We shall see that there are $2^{\aleph_0}$ varieties of De Morgan monoids satisfying these conditions. We provide examples which show that neither of the two hypotheses in Theorem 1 can be dropped. In particular, there are $2^{\aleph_0}$ negatively generated varieties with infinite depth that do not have the ES property, despite the fact that they satisfy a 'weak' version of the ES property.

The second set of sufficient conditions concerns semilinear algebras. A De Morgan monoid is *semilinear* if it is a subdirect product of totally ordered algebras.

**Theorem 2.** *Every variety of negatively generated semilinear De Morgan monoids has the ES property.*

This result uses a combination of representation theorems from [4, 6, 7, 9], and we show that the class of negatively generated semilinear De Morgan monoids is a variety, in fact a locally finite one. Note that Theorem 2 generalizes the earlier finding in [2] that epimorphisms are surjective in every variety of *idempotent* ($x = x \cdot x$) De Morgan monoids, since idempotent De Morgan monoids (a.k.a. Sugihara monoids) are known to be semilinear and negatively generated.

Finally, we will give isolated examples to illustrate that Theorems 1 and 2 do not however encompass all varieties of De Morgan monoids with surjective epimorphisms. These examples will demonstrate that the property of having surjective epimorphisms is not inherited by subvarieties, while the conditions of Theorems 1 and 2 are hereditary. The main results are therefore laborsaving.

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# The Monotone-Light Factorization for 2-categories via 2-preorders

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It is shown that the reflection $2Cat \to 2Preord$ of the category of all 2-categories into the category of 2-preorders determines a monotone-light factorization system on $2Cat$ and that the light morphisms are precisely the 2-functors faithful on 2-cells with respect to the vertical structure. In order to achieve such result it was also proved that the reflection $2Cat \to 2Preord$ has stable units, a stronger condition than admissibility in categorical Galois theory, and that the 2-functors surjective both on horizontally and on vertically composable triples of 2-cells are effective descent morphisms in $2Cat$.

Every map $\alpha : A \to B$ of compact Hausdorff spaces has a factorization $\alpha = me$ such that $m : C \to B$ has totally disconnected fibres and $e : A \to C$ has only connected ones. This is known as the classical monotone-light factorization of S. Eilenberg [3] and G. T. Whyburn [7].

Consider now, for an arbitrary functor $\alpha : A \to B$, the factorization $\alpha = me$ such that $m$ is a faithful functor and $e$ is a full functor bijective on objects. This familiar factorization for categories is as well monotone-light. Meaning that both factorizations are special and very similar cases of categorical monotone-light factorization in an abstract category $\mathbb{C}$, with respect to a full reflective subcategory $\mathbb{X}$, as was studied in [1]. What we shall show is that there is also a monotone-light factorization for 2-categories, very similar to the one given before for categories if one ignores the horizontal composition of 2-cells.

It is well known that any full reflective subcategory $\mathbb{X}$ of a category $\mathbb{C}$ gives rise, under mild conditions, to a factorization system $(\mathcal{E}, \mathcal{M})$. Hence, each of the three reflections $CompHaus \to Prof$, of compact Hausdorff spaces into Stone(profinite) spaces, $Cat \to Preord$, of categories into preorders, and now $2Cat \to 2Preord$, of 2-categories into 2-preorders yields its own reflective factorization system.

Moreover, the process of simultaneously stabilizing $\mathcal{E}$ and localizing $\mathcal{M}$, in the sense of [1], was already known to produce a new non reflective and stable factorization system $(\mathcal{E}', \mathcal{M}^*)$ for the adjunctions $CompHaus \to Prof$ and $Cat \to Preord$. Which is just the (**Monotone**, **Light**)-factorization mentioned above. But this process does not work in general, being the monotone-light factorizations for the reflections $CompHaus \to Prof$ and $Cat \to Preord$ two rare examples. Nevertheless, we shall prove that the (**Full on 2-Cells and Bijective on Objects and Morphisms**, **Faithful on 2-Cells**)-factorization (notice that “full” and “faithful” here are with respect to the vertical composition) for 2-categories is another instance of a successful simultaneous stabilization and localization.

What guarantees the success is the following pair of conditions, which hold in the three cases:

1. the reflection $I : \mathbb{C} \to \mathbb{X}$ has stable units (in the sense of [2]);

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2. for each object $B$ in $\mathbb{C}$, there is a monadic extension (it is said that $(E, p)$ is a monadic extension of $B$, or that $p$ is an effective descent morphism, if the pullback functor $p^* : \mathbb{C}/B \to \mathbb{C}/E$ is monadic) $(E, p)$ of $B$ such that $E$ is in the full subcategory $\mathbb{X}$.

Indeed, the two conditions (1) and (2) trivially imply that the $(\mathcal{E}, \mathcal{M})$-factorization is locally stable, which is a necessary and sufficient condition for $(\mathcal{E}^{'}, \mathcal{M}^*)$ to be a factorization system (see the central result of [1]).

The three reflections may be considered as admissible Galois structures (in which all morphisms are considered), in the sense of categorical Galois theory, since having stable units implies admissibility. Therefore, in the three cases, for every object $B$ in $\mathbb{C}$, we know that the full subcategory $TrivCov(B)$ of $\mathbb{C}/B$, determined by the trivial coverings of $B$ (i.e., the morphisms over $B$ in $\mathcal{M}$), is equivalent to $\mathbb{X}/I(B)$. Moreover, the categorical form of the fundamental theorem of Galois theory gives us even more information on each $\mathbb{C}/B$ using the subcategory $\mathbb{X}$. It states that the full subcategory $Spl(E, p)$ of $\mathbb{C}/B$, determined by the morphisms split by the monadic extension $(E, p)$ of $B$, is equivalent to the category $\mathbb{X}^{Gal(E,p)}$ of internal actions of the Galois pregroupoid of $(E, p)$.

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# The small index property of the Fraïssé limit of finite Heyting algebras

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Consider a countable class of isomorphism types of finitely generated structures with the amalgamation property, the joint embedding property, and the hereditary property. By the *Fraïssé limit* of such a class $\mathcal{K}$, we mean the unique countable structure $M$ whose age, i.e., the class of finitely generated substructures of $M$, is $\mathcal{K}$ up to isomorphism.

One pervasively studied aspect of ultrahomogenous structures—the Fraïssé limit of some classes of structures—is their automorphism groups (see, e.g., Macpherson [3]). Many studies on the automorphism groups of concrete ultrahomogeneous structures involved uniformly locally finite ones, which are necessarily $\omega$-categorical. (For instance, the simplicity of the automorphism group of the countable atomless Boolean algebra, which is ultrahomogeneous and uniformly locally finite, was established by Anderson [1].) The present author offered in an article under review a case study on the automorphism group of a natural non-uniformly locally ultrahomogeneous structure: the Fraïssé limit $L$ of finite Heyting algebras, whose existence follows from Maksimova's result [4] on the Craig interpolation theorem for intuitionistic logic. One main result there was that $\mathrm{Aut}(L)$ was simple.

In the present work, we show yet another important property of $\mathrm{Aut}(L)$. We equip $\mathrm{Aut}(M)$ for an arbitrary countable structure $M$ with the so-called pointwise convergence topology, which is the topology induced as a subset of the Baire space ${}^\omega\omega$ if the domain of $L$ is $\omega$. Under this topology, every open subgroup of $\mathrm{Aut}(M)$, which is now a topological group, has countable indices. With this in mind, a topological group $G$ is said to have the *small index property* if, conversely, every subgroup of $G$ with a countable index is open. The topology of $\mathrm{Aut}(M)$ with the small index property is, therefore, completely determined just from its abstract group structure.

The small index property of $\mathrm{Aut}(M)$ has been shown for many ultrahomogeneous structures $M$. Examples relevant to the present conference include the countable atomless Boolean algebra [5] and the Fraïssé limit of finite distributive lattices [2]. By using the simplicity of $\mathrm{Aut}(L)$ and adapting Truss's argument, we obtain the following:

**Theorem.** The topological group $\mathrm{Aut}(L)$ has the small index property.

Following Truss, we prove this by studying the action of the automorphism group on the dual topological space of the structure. In our case, this space will be an Esakia space. Unlike the case of the countable atomless Boolean algebra, this action will not be transitive; the proof must take care of this appropriately.

Finally, we can further show the *strong* small index property, which has model-theoretic consequences on $L$.

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# Central elements and the Gaeta topos: An algebraic and functorial overview on coextensive varieties

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**Abstract**

We use the theory of central elements to establish a criterion for fp-coextensive varieties that allows to decide whether the Gaeta topos classifies indecomposable objects in terms of the indecomposability of the free algebra on one generator.

In [3] and [5] Lawvere and Schanuel introduced extensive categories as categories $\mathsf{C}$ with finite coproducts and pullbacks in which the canonical functor $+ : \mathsf{C}/X \times \mathsf{C}/X \to \mathsf{C}/(X+Y)$ is an equivalence. Later, in [1] an equivalent and more intuitive description was provided, namely: $\mathsf{C}$ is extensive if and only if it has pullbacks along injections and (finite) coproducts are disjoint and universal [1]. Such a description allowed to visualize more clearly that extensivity is a property typical of categories of "spaces" (e.g., sets, (pre)toposes, topological spaces, compact Hausdorff spaces, Stone spaces, etc.). A category is called *coextensive* if its opposite is extensive. In particular, if $\mathcal{V}$ is a variety (of universal algebras) then we say that $\mathcal{V}$ is coextensive if as an algebraic category it is coextensive. Coextensive varieties are of interest because according to [2] and more recently [4], they could bring an appropriate setting to develop algebraic geometry. Classical examples of coextensive varieties are commutative rings with unit and bounded distributive lattices. Nevertheless, there are a lot of algebraic varieties associated to non-classical logics (as well as classic logic) which are coextensive. This is the case of Boolean Algebras, Heyting algebras, MV-algebras, Gödel algebras and commutative and integral residuated lattices, to name a few. It is also worth mentioning that all these varieties are *varieties with* $\vec{0}$ *and* $\vec{1}$ (i.e. varieties in which there is a positive number $N$ and 0-ary terms $0_1, \ldots, 0_N, 1_1, \ldots, 1_N$ such that $\mathcal{V} \models \bigwedge_{i=0}^{N} 0_i \approx 1_i \Longrightarrow x \approx y$) and that in all these cases, the proof of its coextensivity relies on a suitable description of those elements which concentrate all the information about direct product decompositions, namely the *central elements*. We recall [6] that given a variety $\mathcal{V}$ with $\vec{0}$ and $\vec{1}$ and $\mathbf{A} \in \mathcal{V}$, then $\vec{e} = (e_1, ..., e_N) \in A^N$ is a *central element* of $\mathbf{A}$ if there exists an isomorphism $\tau : \mathbf{A} \to \mathbf{A}_1 \times \mathbf{A}_2$, such that $\tau(e_i) = (0_i^{\mathbf{A}_1}, 1_i^{\mathbf{A}_2})$ for every $1 \leq i \leq N$.

If $\mathsf{C}$ is a small extensive category, the *Gaeta Topology* on $\mathsf{C}$ is the (Grothendieck) topology $J_{\mathcal{G}}$ generated by all finite families $X_i \to X$, such that $\Sigma X_i \to X$ is an isomorphism. The *Gaeta topos* $\mathcal{G}(\mathsf{C})$, is the topos of sheaves on the site $(\mathsf{C}, J_{\mathcal{G}})$. In concrete examples ([7], [4]), it has been proved that the Gaeta topos is the classifying topos of the theory of "connected" objects, which can be considered as the ones who does not admit non-trivial binary coproduct decompositions. Naturally, when considering coextensive categories, the Gaeta topology and the Gaeta topos are related with decompositions into finite products and "indecomposable" (by

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direct-products) objects. That is to say, indecomposable objects are precisely the connected objects in the opposite category. However, in the practice it seems not quite easy to provide an axiomatization of the theory of indecomposable objects when regarding varieties in a more general setting. Fortunately, with the aid of the theory of central elements [6] this obstacle can be tackled. In this context one may be tempted to study those coextensive varieties such that the Gaeta topos classifies the theory of indecomposable objects. As a first step we restrict the problem to those coextensive varieties $\mathcal{V}$ such that the full subcategory of finitely presented algebras of $\mathcal{V}$, $\mathsf{Mod}_{\mathsf{fp}}(\mathcal{V})$ is coextensive. We call such varieties *fp-coextensive*. Moreover, if $\mathcal{V}$ is fp-coextensive, we write $\mathcal{G}(\mathcal{V})$ for the Gaeta topos determined by the extensive category $\mathsf{Mod}_{\mathsf{fp}}(\mathcal{V})^{\mathsf{op}}$.

By using the characterization of coextensive varieties presented in [8], in this talk we provide an algebraic description of those fp-coextensive varieties $\mathcal{V}$ such that $\mathcal{G}(\mathcal{V})$ classifies the theory of $\mathcal{V}$-indecomposable objects by means of the behavior of the free algebra of $\mathcal{V}$ on one generator $\mathbf{F}_{\mathcal{V}}(x)$. Concretely, we will prove the following:

**Theorem 1.** Let $\mathcal{V}$ be a fp-coextensive variety. Then, the following are equivalent:

(1) $\mathcal{G}(\mathcal{V})$ is a classifying topos for $\mathcal{V}$-indecomposable objects.

(2) $\mathbf{F}_{\mathcal{V}}(x)$ is indecomposable in $\mathsf{Set}$.

Afterwards, we apply this result to decide whether the Gaeta topos classifies indecomposable objects in some particular classes of algebras associated to non-classical logics, namely Bounded distributive lattices, Heyting algebras, Gödel algebras and MV-algebras. Some of these results are known and some others are new.

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