Are finite affine topological spaces worthy of study?

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Abstract

Motivated by the theorem of S. A. Morris stating that every topological space is homeomorphic to a subspace of a product of a *finite* (3-element) topological space, the talk shows that this result is no longer valid in case of affine topological spaces (inspired by the concept of *affine set* of Y. Diers), which include, e.g., many-valued topology. In particular, we present an affine analogue of the original 3-element space and show its relationship to an affine analogue of the well-known Sierpinski space, both of which are (in general) infinite. Our message is that finite spaces play a (probably) less important role in affine topological setting (e.g., in many-valued topology) than they do in the classical topology.

1 Finite topological spaces

There exists a well-known result of S. A. Morris [10], stating that every topological space is homeomorphic to a subspace of a product of copies of the *Davey topological space* (or just *Davey space* in this talk) in the terminology of [10], which is a space $\mathcal{D} = (D, \tau_D)$ having a 3-element underlying set $D = \{0, 1, 2\}$ equipped with a topology $\tau_D = \{\emptyset, \{1\}, \{0, 1, 2\}\}$. Stating differently, $\mathcal{D}$ is an extremal coseparator in the category **Top** of topological spaces and continuous maps [1]. In view of this result and to answer the criticism of some researchers claiming that "finite topological spaces are not in the slightest bit interesting", it is stated in [10] that "perhaps there is something of interest in finite spaces after all".

There is another point supporting a general interest in finite topological spaces, i.e., the concept of *Sierpinski space* $\mathcal{S} = (\{0, 1\}, \{\emptyset, \{1\}, \{0, 1\}\})$. This concept plays an important role in general topology. One of its simple but crucial properties is the fact that a topological space is T_0 if and only if it can be embedded into a power of $\mathcal{S}$. Stating differently, $\mathcal{S}$ is an $\mathcal{M}$-coseparator in the category $\mathbf{Top}_0$ of T_0 topological spaces, where $\mathcal{M}$ is the class of topological embeddings (i.e., initial injective maps) in $\mathbf{Top}_0$. Motivated by the importance of the notion of Sierpinski space, E. G. Manes introduced its analogue for concrete categories under the name of *Sierpinski object* [9]. Restated in the language of [1], an object S of a concrete category $\mathbf{C}$ is a *Sierpinski object* provided that for every $\mathbf{C}$-object C, the hom-set $\mathbf{C}(C, S)$ is an initial source.

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As a recent development, [6, 7] showed that finite topological spaces are precisely the *finite preorders* (i.e., finite sets equipped with a reflexive and transitive binary relation). It appeared that nearly all the results of topological descent theory could be motivated by their finite instances, which became simple and natural when expressed in the language of finite preorders.

2 Affine topological spaces

Induced by a considerable number of different approaches to *lattice-valued* (or *many-valued*) topology and a clear lack of intercommunication means between them, an affine approach to lattice-valued topology has been introduced in [11], taking its origin in the notion of *affine set* of Y. Diers [4]. More precisely, while a classical topological space (X, τ) consists of a set X equipped with a topology τ, which, being a subset of the powerset $\mathcal{P}X$ of X, has the algebraic structure of *frame* [8], the affine approach replaces the standard contravariant powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{CBAlg}^{op}$ from the category **Set** of sets to the dual of the category of complete Boolean algebras with a functor $T : \mathbf{X} \to \mathbf{A}^{op}$ from a category $\mathbf{X}$ to the dual category of a variety of algebras $\mathbf{A}$, and requires τ to be a subalgebra of TX. Taking a suitable variety $\mathbf{A}$ and an appropriate functor T, one obtains not only the classical topological spaces, but also, e.g., the closure spaces of [2] as well as the most essential lattice-valued topological frameworks.

This talk tries to investigate the role of *finite* spaces in affine topology. More precisely, there already exists an affine analogue of the Sierpinski space in terms of the Sierpinski object of E. G. Manes [3, 5, 11], which (in general) is no longer finite. This talk provides an affine analogue of the Davey space and shows its simple relation to the affine Sierpinski space. Since the affine Davey space is (in general) no longer finite as well, the simple message we want to convey here is that finite spaces play a (probably) less important role in affine topological setting (for example, in lattice-valued topology) than they do in the classical topology.

References

[1] Adámek, J., Herrlich, H., Strecker, G.E.: Abstract and Concrete Categories: the Joy of Cats. Repr. Theory Appl. Categ. 17, 1–507 (2006)

[2] Aerts, D., Colebunders, E., van der Voorde, A., van Steirteghem, B.: State property systems and closure spaces: a study of categorical equivalence. Int. J. Theor. Phys. 38(1), 359–385 (1999)

[3] Denniston, J.T., Melton, A., Rodabaugh, S.E., Solovyov, S.: Sierpinski object for affine systems. Fuzzy Sets Syst. 313, 75–92 (2017)

[4] Diers, Y.: Categories of algebraic sets. Appl. Categ. Struct. 4(2-3), 329–341 (1996)

[5] Giuli, E., Hofmann, D.: Affine sets: the structure of complete objects and duality. Topology Appl. 156(12), 2129–2136 (2009)

[6] Janelidze, G., Sobral, M.: Finite preorders and topological descent I. J. Pure Appl. Algebra 175(1-3), 187–205 (2002)

[7] Janelidze, G., Sobral, M.: Finite preorders and topological descent II: étale descent. J. Pure Appl. Algebra 174(3), 303–309 (2002)

[8] Johnstone, P.T.: Stone Spaces. Cambridge University Press (1982)

[9] Manes, E.G.: Compact Hausdorff objects. General Topology Appl. 4, 341–360 (1974)

[10] Morris, S.A.: Are finite topological spaces worthy of study? Aust. Math. Soc. Gaz. 11, 31–32 (1984)

[11] Solovyov, S.: Sobriety and spatiality in varieties of algebras. Fuzzy Sets Syst. 159(19), 2567–2585 (2008)