Quantum logics as algebras for monads

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In [7], Kalmbach proved the following theorem.

Theorem 1. *Every bounded lattice L can be embedded into an orthomodular lattice $K(L)$.*

The proof of the theorem is constructive, $K(L)$ is known under the name *Kalmbach extension* or *Kalmbach embedding*. In [8], Mayet and Navara proved that Theorem 1 can be generalized: every bounded poset P can be embedded into an orthomodular poset $K(P)$. In 2004 Harding explained where does the Kalmbach construction come from:

Theorem 2. *[3, Theorem 16] K is a functor left adjoint to the forgetful functor from the category of orthomodular posets to the category of bounded posets.*

However, as remarked by Harding in the same paper, K does not restrict to a functor between the category of orthomodular lattices and the category of bounded lattices. As every adjunction, the adjunction from Theorem 2 induces a monad on the category of orthomodular posets, which we call *the Kalmbach monad*.

In their seminal paper [2], Foulis and Bennett introduced the notion of an *effect algebra*.

Theorem 3. *[6] The category of effect algebras is equivalent to the category of algebras for the Kalmbach monad.*

In other words, the forgetful functor from the category of effect algebras to the category of bounded posets in monadic. This theorem means the category of effect algebras in inherently present in the forgetful functor from orthomodular posets to bounded posets.

In [1], Dvurečenskij and Vetterlein introduced a non-commutative generalization of effect algebras, called *pseudo-effect algebras*.

Theorem 4. *[4] The forgetful functor from the category of pseudo effect algebras to the category of bounded posets is monadic.*

An important subcategory of the effect algebras is the category of *ω-effect algebras*, in which sums of certain countable families of elements are required to exist.

Theorem 5. *[9] The forgetful functor from the category of ω-effect algebras to the category of bounded posets is monadic.*

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Unlike the other adjunctions here, the adjunction between orthomodular posets and bounded posets from Theorem 2 is not monadic. This leads to a natural question: is the category of orthomodular posets isomorphic to a category of algebras for a monad on some category, in a nontrivial way? The following theorem answers this questions in the positive.

Theorem 6. *[5] The forgetful functor from the category of orthomodular posets to the category of bounded posets with involution is monadic.*

The proofs of the right-adjointness of the forgetful functor in Theorems 4, 5 and 6 use the general adjoint functor theorem, hence these proofs are non-constructive.

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