Hausdorff Polynomial Functors

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We study endofunctors on categories of metric spaces analogous to the Kripke polynomial set functors. The latter form a 'well-behaved' class of functors F including a number of examples where F-coalgebras are well-known state-based systems. (For example Kripke structures.) Kripke polynomial functors are the set functors F obtained from Id, the finite power-set functor $\mathcal{P}_f$ and the constant functors A (A any set) by using products, coproducts, and composition:

$$F ::= \mathcal{P}_f \,|\, \mathrm{Id} \,|A| \prod_{i\in I} F_i \mid \coprod_{i\in I} F_i | FF$$

(Various other versions are used in the literature, restricting e.g. coproducts to finite ones etc.)

It follows from the results of Worrell [7] that each such functor F has a terminal coalgebra obtained in $\omega+\omega$ steps of the construction of terminal coalgebras introduced (in dual form) in [1]. This transfinite construction is given by $V_0 = 1$, the terminal object, $V_{i+1} = FV_i$ for every $i < \omega+\omega$, and V_ω is the limit of the ω^{op}-chain V_i ($i < \omega$). For every Kripke polynomial functor F the terminal coalgebra is the limit $\lim\limits_{i<\omega+\omega} V_i$. Consequently, every Kripke polynomial functor F generates a cofree comonad $\overline{F}$ obtained as a limit of the chain $\overline{V}_i$ ($i < \omega+\omega$) of endofunctors defined by $\overline{V}_0 = \mathrm{Id}$, $\overline{V}_{i+1} = F\cdot\overline{V}_i + \mathrm{Id}$ and $\overline{V}_\omega = \lim\limits_{i<\omega} \overline{V}_i$: we have $\overline{F} = \lim\limits_{i<\omega+\omega} \overline{V}_i$.

Recall that $\mathcal{P}_f$ is the monad on **Set** of semilattices with zero.

For coalgebraic modal logic the Vietoris endofunctor on the category of Stone spaces (assigning to a space all compact subsets with the Vietoris topology) plays an important role. This led Kurz et al. [5] to introduce Vietoris polynomial functors by the analogous grammar to above: just $\mathcal{P}_f$ is substituted by the Vietoris functor.

We now consider similar collections of endofunctors on the category **Met** of extended metric spaces (that is, distance ∞ is admitted) and nonexpanding maps. And on its full subcategories **CMet** of complete spaces and **KMet** of compact spaces. The role of $\mathcal{P}_f$ is played be the *Hausdorff functor* assingning to a space X the Hausdorff space $\mathcal{H}X$ of all nonempty compact subsets with the Hausdorff metric

$$d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$$

where $d(a,B) = \inf_{b\in B} d(a,b)$. The functor $\mathcal{H}$ preserves completeness, and it thus yields an endofunctor of **CMet**, also denoted by $\mathcal{H}$. This is the semilattice monad on **CMet**, as proved in [3]. (Surprisingly, the functor $\mathcal{H}$ is finitary on **CMet**, see [2].)

Analogously, the semilattice monad on **Met** assigns to X the space $\mathcal{H}_f X$ of all nonempty finite subsets with the Hausdorff metric, as proved in [6].

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Definition. *Hausdorff polynomial functors* on **CMet** are the endofunctors F obtained from the Hausdorff functor, Id and the constant functors A (any object) by using products, coproducts, and composition:

$$F ::= \mathcal{H} | \operatorname{Id} | A | \prod_{i \in I} F_i \mid \coprod_{i \in I} F_i | FF$$

We now obtain a result about cofree comonads for the above functors analogous to Kripke polynomial functors. An *isometric embedding* is a morphism of **Met** preserving distances. A cone $f_i \colon A \to A_i$ $(i \in I)$ is called *isometric* if $\langle f_i \rangle \colon A \to \prod_{i \in I} A_i$ is an isometric embedding.

Theorem. *The Hausdorff functor $\mathcal{H}$ on* **CMet** *preserves isometric embeddings and their wide intersections. Moreover, it preserves isometric cones of ω^{op}-chains.*

Corollary. Every Hausdorff polynomial functors F on **CMet**
(1) has a terminal coalgebra obtained in $\omega + \omega$ steps: $\nu F = \lim_{i < \omega + \omega} V_i$, and
(2) generates a cofree comonad in $\omega + \omega$ steps: $\overline{F} = \lim_{i < \omega + \omega} \overline{V}_i$.

It follows that the category of coalgebras for F is complete.

An analogous corollary holds for **Met**, where the polynomial functors are as above with $\mathcal{H}_f$ replacing $\mathcal{H}$. (Actually, that corollary also holds for **Met** *without* this replacement: every endofunctor on **Met** as in the above definition has a terminal coalgebra obtained in $\omega + \omega$ steps.)

The situation with **KMet** is completely different: as demonstrated by Hofmann and Nora [4], the Hausdorff polynomial functor $\mathcal{H} + 1$ on this subcategory does not have a terminal coalgebra. In op.cit. the name Hausdorff polynomial functors is used in a different manner: the elements of $\mathcal{H}X$ are the closed subsets of X rather than the compact ones. (On the category **KMet** this makes no difference, of course.) Terminal coalgebras for such endofunctors on **CMet** are proved not to exist in op.cit.

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