Deduction via 2-category theory

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The purpose of this talk is to present a category-based unified approach that accommodates diverse takes on the topic of *deduction*. The effort required in order to do so turns out to be extremely fruitful, and in fact it can be used, for example, to obtain novel results about the algebraic treatment of type constructors in dependent type theory.

One of the motivating examples is to give a theoretical framework in which the two following rules, which stand on very conceptually different grounds, can be compared.

$$\text{(Subs)}\ \frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B}{\Gamma \vdash B[a]} \qquad\qquad \text{(Cut)}\ \frac{x;\Gamma \vdash \phi \qquad x;\Gamma,\phi \vdash \psi}{x;\Gamma \vdash \psi}$$

One can traditionally be found in type theory [6], the other in proof theory [8]: despite their incredibly similar look, and the somehow parallel development of the respective theories in the same notational framework, there are some philosophical differences between the interpretation of the symbols above. Not only that, but the same "$\vdash$" symbol seems to regard only statements of one kind `formula` in the case of (Cut), while it pertains to two - `term` and `type` - in that of (Subs).

Of course one could argue that these different points of view are mostly philosophical, and, in particular, the deep connection between proof theory and type theory has been studied for a while: its development falls under the paradigm that is now mostly known as *propositions-as-types* [9]. We believe our theory gives testament to that and, in fact, it gives it a categorical backbone.

Rebooting some ideas from [5], we develop what we call *judgemental theories*. Going back to the example of (Subs) and (Cut), we intuitively see how they both fit the same paradigm, in the sense that we could read both as instances of the following syntactic string of symbols

$$(\triangle)\ \frac{\heartsuit\ \vdash \blacksquare \qquad \square \vdash \clubsuit}{\heartsuit\ \vdash \spadesuit}$$

which we usually parse as: *by* $\triangle$, *given* $\heartsuit\ \vdash \blacksquare$ *and* $\square \vdash \clubsuit$ *we deduce* $\heartsuit\ \vdash \spadesuit$. Our theory allows for a coherent expression of all such strings of symbols, and shows how a suitable choice of *context* either produces (Subs) or (Cut): it is not about the interpretation of the symbols, just about the relation they are in with one another.

Concretely, a judgemental theory is a 2-subcategory of **Cat** closed under some constructions which aim to encode deductive power into the system, such as finite limits and lifting of 2-cells along fibrations, but everything that we develop can be inherently repeated into any 2-category. Each "kind" of entailment/context relation ($\vdash$) is represented by a functor - often, a fibration - over a fixed context category. Each rule is represented by a (lax) commutative triangle - often, a morphism of fibrations - involving such functors. Starting from a bunch of such choices, we show that a few categorical constructions allow us to produce new (lax) commutative triangles,

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hence new rules. In fact, they produce all structural rules both in the case of dependent type theory and of natural deduction.

Being very syntactic in nature, our framework has both the advantage of being versatile and computationally meaningful. It allows, for example, to give a general definition of type constructor, a feature that has not been available before.

If the process of formalization of a given deductive system is purely syntactical, in the sense that we are not interested in what a given judgement or rule should *mean*, only in the symbols involved, the judgemental theory we obtain is often as well behaved as one hopes a categorical semantics would be: we consider the case study of dependent types, and show how traditional categorical models ([5], [7], [4], [3], [2]) all fit into our paradigm. Moreover, properties that were considered *external*, such has having dependent sums for CE-systems [1], are *internalized* in our framework, so that one can quantitatively compare different models.

References

[1] Ahrens, Benedikt and Emmenegger, Jacopo and Randall North, Paige and Rijke, Egbert (2021) *B-systems and C-systems are equivalent*, under review

[2] Awodey, Steve (2018) *Natural models of homotopy type theory*, Mathematical Structures in Computer Science, vol. 28

[3] Cartmell, John (1986) *Generalised algebraic theories and contextual categories*, Annals of Pure and Applied Logic, vol. 32

[4] Dybjer, Peter (1996) *Internal type theory*, Springer Berlin Heidelberg, pp. 120-134

[5] Jacobs, Bart (1999) *Categorical logic and type theory*, Elsevier

[6] Martin-Löf, Per and Sambin, Giovanni (1984) *Intuitionistic type theory*, Bibliopolis Naples, vol. 9

[7] Taylor, Paul (1999) *Practical Foundations of Mathematics*, Cambridge Studies in Advanced Mathematics, vol. 59

[8] Troelstra, Anne Sjerp and Schwichtenberg, Helmut (2000) *Basic proof theory*, Cambridge University Press, vol. 43

[9] Wadler, Philip (2015) *Propositions as types*, Communications of the ACM, vol. 58