Projectivity in (bounded) commutative integral residuated lattices

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We approach the study of projective algebras in varieties of (bounded) commutative integral residuated lattices. Our point of view is going to be algebraic, however projectivity is a categorical concept, and therefore our findings pertain to the corresponding algebraic categories as well. Being projective in a variety of algebras, or in any class containing all of its free objects, corresponds to being a retract of a free algebra, and projective algebras contain relevant information both on their variety and on its lattice of subvarieties.

In particular, as first noticed by McKenzie [8], there is a close connection between projective algebras in a variety and *splitting algebras* in its lattice of subvarieties. The notion of splitting algebra comes from lattice theory, and studying splitting algebras is particularly useful to understand lattices of subvarieties, since a splitting algebra divides a subvariety lattice in a disjoint union of a principal filter defined by its generated variety, and a principal ideal.

The varieties of algebras that are the object of our study are relevant both in the realm of algebraic logic and from a purely algebraic point of view. In fact, residuated structures arise naturally in the study of many interesting algebraic systems, such as ideals of rings or lattice-ordered groups, besides encompassing the equivalent algebraic semantics (in the sense of [2]) of substructural logics. We refer the reader to [6] for detailed information on this topic. The Blok-Pigozzi notion of algebraizability entails that the logical deducibility relation is fully and faithfully represented by the algebraic equational consequence of the corresponding algebraic semantics, and therefore logical properties can be studied algebraically, and viceversa. Substructural logics are a large framework and include most of the interesting non-classical logics: intuitionistic logics, relevance logics, and fuzzy logics to name a few, besides including classical logic as a special case. Therefore, substructural logics on one side, and residuated lattices on the other, constitute a wide unifying framework in which very different structures can be studied uniformly.

The investigation of projective structures in particular varieties of residuated lattices has been approached by several authors (see for instance [1],[3], [4], [5], [7]). However, to the best of our knowledge, no effort has yet been done to provide a uniform approach in a wider framework, which is what we attempt to start here.

Besides some general findings on FL_{ew}-algebras, our main results concern varieties with particular properties: varieties closed with respect to the ordinal sum construction, and varieties where the lattice order is actually the inverse divisibility ordering. With these methods, we show that several interesting varieties in the realm of algebraic logic have the property that every finitely presented algebra is projective, among which: locally finite varieties of hoops and bounded hoops, and cancellative hoops. As a consequence of some general results about

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projectivity in varieties closed under ordinal sums, we also show an alternative proof of the characterization of finite projective Heyting algebras.

In the more general setting, via the connection with splitting algebras, we show that the only finite projective algebra in FL_{ew} is the two elements Boolean algebra **2**, while we identify a large class of structures where all finite Boolean algebras are projective.

Interestingly, the study of projective algebras in this realm has a relevant logical application. Indeed, following the work of Ghilardi [7], the study of projective algebras in a variety is strictly related to unification problems for the corresponding logic. More precisely, solving a unification problem is equivalent to finding a homomorphism from a suitable finitely presented algebra **A** in a projective algebra, and if this is possible then **A** is said to be unifiable. So the case in which a finitely presented algebra is unifiable if and only if it is projective (as is the case in the examples quoted above) is noteworthy in unification theory. We will illustrate some immediate consequences of this connection.

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