Semilinear idempotent distributive ℓ-monoids

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A *distributive ℓ-monoid* is an algebra $\mathbf{M} = \langle M, \cdot, \wedge, \vee, e\rangle$ such that $\langle M, \cdot, e\rangle$ is a monoid, $\langle M, \wedge, \vee\rangle$ is a distributive lattice, and for all $a, b, c, d \in M$

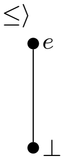
$$a(b \wedge c)d = abd \wedge acd \quad \text{and} \quad a(b \vee c)d = abd \vee acd.$$

The class $\mathcal{DLM}$ of distributive ℓ-monoids forms a variety (equational class). We call a distributive ℓ-monoid *idempotent* or *commutative* if its monoid reduct is idempotent or commutative, respectively. A distributive ℓ-monoid is called *semilinear* if it is isomorphic to a subdirect product of totally ordered monoids, i.e., distributive ℓ-monoids where the lattice reduct is a total order. For a totally ordered monoid $\mathbf{M}$, we write $\mathbf{M} = \langle M, \cdot, e, \leq\rangle$, where $\leq$ is the lattice-order of $\mathbf{M}$. The class $\mathcal{SemDLM}$ of semilinear distributive ℓ-monoids forms a variety that is generated by the class of totally ordered monoids and every subdirectly irreducible member of this variety is totally ordered. Moreover it is shown in [3] (see also [1]) that every commutative distributive ℓ-monoid is semilinear.

The aim of this work is to study the variety $\mathcal{SemIdDLM}$ of semilinear idempotent distributive ℓ-monoids and its subvariety $\mathcal{CIdDLM}$ of commutative idempotent distributive ℓ-monoids. Bearing in mind that $\mathcal{SemIdDLM}$ is locally finite, we use the e-sum construction of [4,5] (see also [2]) to investigate the structure of finite totally ordered idempotent monoids. Let $\mathbf{L} = \langle L, \cdot_L, e, \leq_L\rangle$ and $\mathbf{M} = \langle M, \cdot_M, e, \leq_M\rangle$ be totally ordered idempotent monoids, where we relabel the elements of M and L such that $M \cap L = \{e\}$. Then the *e-sum* of $\mathbf{L}$ and $\mathbf{M}$ is defined as $\mathbf{L} \oplus \mathbf{M} = \langle M \cup L, \cdot, e, \leq\rangle$, where $\cdot$ is the extension of the monoid operations $\cdot_L$ and $\cdot_M$ with $a \cdot b = b \cdot a = a$ for all $a \in L \setminus \{e\}$ and $b \in M$ and $\leq$ is the least extension of the orders $\leq_L$ and $\leq_M$ to $L \cup M$ that satisfies for all $a \in L \setminus \{e\}$, $b \in M$ that $a \leq b$ if $a \leq_L e$ and $b \leq a$ if $e \leq_L a$. The e-sum of two totally ordered idempotent monoids is again a totally ordered idempotent monoid and the operation of taking e-sums is associative. Accordingly we write $\bigoplus_{i=1}^{n} \mathbf{M}_i$ for $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_n$, where $\bigoplus_{i=1}^{0} \mathbf{M}_i \coloneqq \mathbf{0}$ is a trivial algebra. It turns out that every finite totally ordered idempotent monoid can be constructed as an e-sum using only the four algebras $\mathbf{C}_2$, $\mathbf{C}_2^{\partial}$, $\mathbf{G}_3$, and $\mathbf{D}_3$ described below.

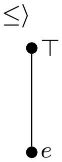
$\mathbf{C}_2 = \langle\{\bot, e\}, \cdot, e, \leq\rangle$

$\cdot$	e	$\bot$
e	e	$\bot$
$\bot$	$\bot$	$\bot$



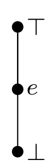
$\mathbf{C}_2^{\partial} = \langle\{e, \top\}, \cdot, e, \leq\rangle$

$\cdot$	e	$\top$
e	e	$\top$
$\top$	$\top$	$\top$



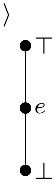
$\mathbf{G}_3 = \langle\{\bot, e, \top\}, \cdot, e, \leq\rangle$

$\cdot$	e	$\bot$	$\top$
e	e	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$



$\mathbf{D}_3 = \langle\{\bot, e, \top\}, \cdot, e, \leq\rangle$

$\cdot$	e	$\bot$	$\top$
e	e	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\top$
$\top$	$\top$	$\bot$	$\top$



Theorem. Every finite totally ordered idempotent monoid is isomorphic to an e-sum $\bigoplus_{i=1}^{n} \mathbf{M}_i$ with $\mathbf{M}_i \in \{\mathbf{C}_2, \mathbf{C}_2^\partial, \mathbf{G}_3, \mathbf{D}_3\}$. Moreover, this e-sum is unique with respect to the algebras $\mathbf{C}_2$, $\mathbf{C}_2^\partial$, $\mathbf{G}_3$, $\mathbf{D}_3$.

We also characterize the finite subdirectly irreducibles of $\mathcal{SemIdDLM}$ in terms of e-sums.

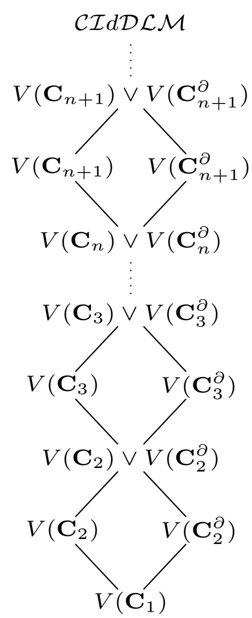
Theorem. A finite totally ordered idempotent ℓ-monoid $\mathbf{M}$ is subdirectly irreducible if and only if there exists an $n > 0$ and algebras $\mathbf{M}_i \in \{\mathbf{C}_2, \mathbf{C}_2^\partial, \mathbf{G}_3, \mathbf{D}_3\}$ for $i \in \{1, \ldots, n\}$ such that $\mathbf{M} \cong \bigoplus_{i=1}^{n} \mathbf{M}_i$ and $\mathbf{M}_i = \mathbf{M}_{i+1}$ implies $\mathbf{M}_i \in \{\mathbf{G}_3, \mathbf{D}_3\}$ for every $i \in \{1, \ldots, n-1\}$.

Using this characterization and [5, Corollary 4.3] we prove:

Theorem. The subvariety lattice of $\mathcal{SemIdDLM}$ is countably infinite.

For the commutative case the characterization of the finite subdirectly irreducibles yields that for every $n > 1$ the variety $\mathcal{CIdDLM}$ contains up to isomorphism exactly two n-element subdirectly irreducibles which we denote by $\mathbf{C}_n$ and $\mathbf{C}_n^\partial$. Setting $\mathbf{C}_1 = \mathbf{C}_1^\partial$ to be a trivial algebra, we can give an explicit characterization of the subvariety lattice of $\mathcal{CIdDLM}$.

Theorem. The subvariety lattice of $\mathcal{CIdDLM}$ is of the following form, where $V(\mathbf{A})$ denotes the variety generated by $\mathbf{A}$:



References

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