Duality, unification, and admissibility in the positive fragment of Łukasiewicz logic

Sara Ugolini[1*]

Artificial Intelligence Research Institute, Spanish National Research Council (IIIA, CSIC)
Bellaterra, Barcelona, Spain
sara@iiia.csic.es

MV-algebras, the equivalent algebraic semantics in the sense of Blok-Pigozzi of Łukasiewicz logic, have a deep connection with interesting geometrical objects. Indeed, the category of finitely presented MV-algebras with homomorphisms is dually equivalent to a category whose objects are rational polyhedra and the morphisms are so-called $\mathbb{Z}$-maps [12]. This connection allows the study of relevant algebraic and logical properties from the geometrical point of view, such as, for instance, the study of projective algebras and amalgamation, or correspondingly, the investigation of interpolation and unification problems [6, 8, 9, 12, 14].

Looking at Łukasiewicz logic as a substructural logic, thus as an axiomatic extension of the Full Lambek Calculus with exchange and weakening [10], we consider its positive (i.e., 0-free) fragment. The latter is also algebraizable, and its corresponding equivalent algebraic semantics is the variety of Wajsberg hoops. Wajsberg hoops are interesting structures also from a purely algebraic point of view. They play an important role in the theory of hoops [4], which are naturally ordered commutative monoids, and they have a particular connection with lattice-ordered abelian groups (abelian ℓ-groups for short). In fact, the variety WH of Wajsberg hoops is generated by its totally ordered members, that are, in loose terms, either negative cones of abelian ℓ-groups, or *intervals* of abelian ℓ-groups [2] (equivalently, MV-algebras, via Mundici's Γ functor [13]). In the context of the algebraic semantics of many-valued logics, the relevance of Wajsberg hoops is also related to the study of the equivalent algebraic semantics of Hájek Basic Logic and its positive subreducts (BL-algebras and basic hoops). Given the well-known decomposition result in terms of Wajsberg hoops for totally ordered BL-algebras given by Aglianò and Montagna [1], the understanding of Wajsberg hoops is key to obtain interesting results in this framework.

We show that finitely presented Wajsberg hoops have an interesting geometrical dual as well, in particular, with what we will call *pointed* rational polyhedra. More precisely, we show how finitely presented Wajsberg hoops are dually equivalent to a (non-full) subcategory of rational polyhedra with $\mathbb{Z}$-maps, given by rational polyhedra in unit cubes $[0,1]^n$ that contain the lattice point $\mathbf{1} = (1,\ldots,1)$, and *pointed* $\mathbb{Z}$-maps, that are $\mathbb{Z}$-maps that respect the lattice point $\mathbf{1}$. In particular, we use a key result in [2] to first show that Wajsberg hoops are equivalent to a (non full) subcategory of finitely presented MV-algebras, and then we suitably restrict the Marra-Spada duality [12].

The connection with the MV-algebraic duality with rational polyhedra allows the use of the deep results obtained by Cabrer and Mundici about finitely generated projective MV-algebras [8, 9, 6] to describe finitely generated projective Wajsberg hoops. In particular, we show that no MV-algebra (or more precisely, its 0-free reduct) is projective in the variety of Wajsberg hoops, and actually that finitely generated nontrivial projective Wajsberg hoops are necessarily unbounded. Interestingly enough, this implies that, in particular, the (0-free reduct of the)

[*]Speaker.

two-element Boolean algebra **2** is not projective in the variety of residuated lattices, while **2** is projective in every variety of bounded commutative integral residuated lattices, and in the variety of all bounded commutative integral residuated lattices it is the only finite projective algebra [3].

The fact that Wajsberg hoops are the equivalent algebraic semantics of the positive fragment of Łukasiewicz logic allows us to use our algebraic and geometric investigation to derive some analogies and differences between Łukasiewicz logic and its positive fragment. In particular, while deducibility in the fragment coincides with deducibility of positive terms in Łukasiewicz logic, the same is not true for admissibility of rules. That is, there are rules involving positive terms that are admissible in the positive fragment but not in Łukasiewicz logic.

Moreover, via the algebraic approach to unification problems developed by Ghilardi [11], we will show that the unification type of the variety of Wajsberg hoops, and thus of the positive fragment of Łukasiewicz logic, is nullary. This is in close analogy with the case of MV-algebras, and indeed our proof adapts to pointed rational polyhedra the pathological example given in [12] for the case of Łukasiewicz logic. Furthermore, via the algebraic approach to admissibility developed in [7], we show that while the exact unification type of Łukasiewicz logic is finitary, the one of its positive fragment is unitary. This in particular implies decidability of the admissibility of rules in Wajsberg hoops and in the positive fragment of Łukasiewicz logic.

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