# Contextuality in logical form: Duality for transitive partial CABAs

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Partial Boolean algebras were introduced by Kochen and Specker in their seminal work on contextuality in quantum mechanics [3, 2], as a natural (algebraic-)logical setting for contextual systems, corresponding to a calculus of partial propositional functions. They provide an alternative to traditional Birkhoff–von Neumann quantum logic [1] in which operations such as conjunction and disjunction are partial, being only defined in the domain where they are physically meaningful. In the key example of the projectors on a Hilbert space, the operations are only defined for commuting projectors, which correspond to properties of the quantum system that can be tested simultaneously.

We extend the classical Lindenbaum–Tarski dualities between finite sets and finite Boolean algebras, and more generally between sets and complete atomic Boolean algebras (CABAs), to the setting of (transitive) partial Boolean algebras. Specifically, we establish a dual equivalence between the category of transitive partial CABAs and a category of exclusivity graphs with an appropriate notion of morphism.

The vertices of an exclusivity graph may be interpreted as *possible worlds* of maximal information, with edges representing logical incompatibility or mutual exclusivity between two worlds. The classical case corresponds to complete graphs, as all possible worlds are mutually exclusive. Similarly, the appropriate notion of morphism is relaxed from functions to certain kinds of relations. From an exclusivity graph, a transitive partial CABA is constructed whose elements are sets of mutually exclusive worlds (cliques of the graph) modulo an equivalence relation. This equivalence identifies cliques that jointly exclude the same set of worlds, i.e. that have the same neighbourhood. The main result shows, in particular, that any transitive partial CABA can be recovered in this fashion from its graph of atoms with the logical exclusivity relation.

We also give an explicit construction of the free transitive partial CABA on a set of propositions with a compatibility relation, via an adjunction between compatibility graphs and exclusivity graphs that generalises the powerset self-adjunction from the classical case.

The duality reveals a connection between the algebraic-logical setting of partial Boolean algebra and the graph-theoretic approach to contextuality of Cabello–Severini–Winter. Under it, a transitive partial CABA witnessing contextuality, in the Kochen–Specker sense that it has no homomorphism to the two-element Boolean algebra, corresponds to a graph with no 'points', i.e. with no maps from the singleton graph, which are in bijection with stable, maximum clique transversal sets.

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# References


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