Unitless Frobenius quantales*

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It is often stated that a Frobenius quantale necessarily is unital. While this is correct if Frobenius quantales are defined starting from a dualizing element, it is also possible to consider negations as primitive operations and axiomatize them so to ensure some coherency w.r.t. implications.

Definition 1. A *Frobenius quantale* is a tuple $(Q, *, {}^{\perp}(-), (-)^{\perp})$ where $(Q, *)$ is a quantale and ${}^{\perp}(-), (-)^{\perp} : Q \longrightarrow Q$ are inverse antitone maps satisfying

$$x \backslash {}^{\perp}y = x^{\perp}/y\,, \qquad \text{for all } x, y \in Q. \tag{1}$$

The map $(-)^{\perp}$ is called the *right negation* while the map ${}^{\perp}(-)$ the *left negation*. A *Girard quantale* is a Frobenius quantale for which right and left negations coincide.

Axiom (1) explicitly appears in [4] and similar (and actually equivalent) relations, such as

$$x\backslash y = x^{\perp}/y^{\perp}\,, \qquad x/y = {}^{\perp}x\backslash {}^{\perp}y\,, \qquad {}^{\perp}x\backslash y = x/y^{\perp}\,.$$

have been pointed out in the literature, see e.g. [2, 9]. Of course, if a quantale Q has a dualizing element 0, then the two negations ${}^{\perp}(-) := 0/-$ and $(-)^{\perp} := -\backslash 0$ satisfy (1). Also, if a Frobenius quantale Q is unital, then the two negations give rise to a dualizing element $1^{\perp} = {}^{\perp}1$, so the previous definition does not yield novelties for unital quantales. According to it, however, we can have Frobenius quantales that are unitless. For example, for a quantale Q, its Chu construction $Chu(Q)$ is a Girard quantale which is unital if and only if Q is unital.

Our aim is to have a first glance on these structures and decide on the worthiness of future research. We firstly observe that the standard representation theory via phase quantales can be lifted to unitless Girard quantales and even to unitless Frobenius quantales.

Definition 2. For a quantale Q, a *Serre*[1] *Galois connection* is a Galois connection on (l, r) on Q such that $l \circ r = r \circ l$ and $x\backslash l(y) = r(x)/y$, for all $x, y \in Q$.

Theorem 3. *If (l, r) is a Serre Galois connection on Q, then $j = r \circ l = l \circ r$ is a nucleus on Q. The quantale of fixed-points of j, Q_j, is then a Frobenius quantale where the left (resp., right) negation is given by the restriction of l (resp., r) to Q_j.*

Every Frobenius quantale arises in this way:

Theorem 4. *If Q is a Frobenius quantale, then the powerset quantale $P(Q)$ has a canonical Serre Galois connection l, r such that, for $j = l \circ r$, the quantale $P(Q)_j$ is isomorphic to Q.*

Motivations and examples for developing this theory stem from the following result:

Theorem 5 (See e.g. [7, 2, 3, 11, 10])**.** *The quantale of sup-preserving endomaps of a complete lattice L is a Frobenius quantale if and only if L is completely distributive.*

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*Full version available as [1].

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[1]The naming originates from [9].

and also from lattice theoretic constructions [12, 5] related to Raney's notion of tight Galois connection [8]. Recall the definition of Raney's transforms:

$$f^{\vee}(x) = \bigvee_{x \not\leq t} f(t)\,, \qquad\qquad g^{\wedge}(x) = \bigwedge_{t \not\leq x} g(t)\,.$$

For L a complete lattice, a sup-preserving preserving map $f : L \longrightarrow L$ is *tight* if $f = f^{\wedge\vee}$. We decompose the sufficient condition of Theorem 5 as follows:

Theorem 6. *The set of tight endomaps of a complete lattice L is a Girard quantale.*

Then, using Raney's characterisation of completely distributive lattices [8], we have:

Theorem 7. *The Girard quantale of tight endomaps of L is unital if and only if L is a completely distributive lattice, if and only if the identity of L is tight, if and only if every sup-preserving endomap of L is tight.*

There is a precise analogy between tight maps and trace class operators on an infinite dimensional Hilbert space H: these are nuclear maps [6] in the appropriate autonomous categories. Let $B_1(H)$ be the ideal of trace class operators: as an algebra, it cannot have a unit. The trace operation allows to define a (self-adjoint) Serre Galois connection (l, l) on the powerset quantale $P(B_1(H))$, where $B_1(H)$ is considered as a monoid w.r.t. multiplication. Letting $j = l^2$ in the next statement, we obtain a generalised version of the Girard quantale of subspaces of a finite dimensional C*-algebra:

Theorem 8. *$P(B_1(H))_j$ is a Girard quantale with no unit.*

It might be thought that some completion precess allows to add units to Frobenius quantales. This is actually true, yet the resulting embedding does not preserve the negations. There is indeed a fundamental obstruction towards adding units:

Theorem 9. *Let Q be a Frobenius quantale for which there exists a quantale embedding into a unital Frobenius quantale which also preserves negations. Then $\bigwedge_{x \in Q} x\backslash x$ is a unit of Q.*

In order to further understand the structure of unitless Frobenius quantales, we have investigated tight endomaps of M_n, the finite modular lattice with n atoms which are also coatoms. We give characterizations of these endomaps and enumerate them. For a tight sup-preserving endomap f of M_n, the implications $f\backslash f$ (one implication computed in the quantale of tight endomaps and the other computed in the quantale of all sup-preserving endomaps) coincide. This ensures reasonable properties of elements of the form $f\backslash f$, for example they are idempotent. It is easily argued, then, that elements of this form are not closed under infima. We do not know yet whether similar phenomena hold for quantales of tight endomaps of L when L is an arbitrary complete lattice.

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