The Dependence Problem in Varieties of Modal Semilattices

Naomi Tokuda

Mathematical Institute, University of Bern, Switzerland
naomi.tokuda@unibe.ch

In [1] De Jongh and Chagrova introduced the notion of dependence for intuitionistic propositional logic IPC. Formulas $\varphi_1, \dots, \varphi_n$ are called IPC-*dependent* if there exists a formula $\psi(p_1, \dots, p_n)$ such that $\vdash_{\mathsf{IPC}} \psi(\varphi_1, \dots, \varphi_n)$ and $\nvdash_{\mathsf{IPC}} \psi(p_1, \dots, p_n)$, otherwise $\varphi_1, \dots, \varphi_n$ are called IPC-*independent*.

In [4] we generalise this notion to a universal algebra setting. Let $\mathcal{L}$ be an algebraic language and let $\mathcal{V}$ be a variety of $\mathcal{L}$-algebras. By $\mathrm{Tm}(\bar{x})$ and $\mathrm{Eq}(\bar{x})$, we denote the set of $\mathcal{L}$-terms and $\mathcal{L}$-equations over the set of variables $\bar{x}$, respectively. We call terms $t_1, \dots, t_n \in \mathrm{Tm}(\bar{x})$ $\mathcal{V}$-*dependent* if for some equation $\varepsilon(y_1, \dots, y_n)$,

$$\mathcal{V} \vDash \varepsilon(t_1, \dots, t_n) \quad \text{and} \quad \mathcal{V} \nvDash \varepsilon;$$

otherwise, we call $t_1, \dots, t_n$ $\mathcal{V}$-*independent*. This notion of dependence is related to a general algebraic notion of dependence introduced by Marcewski in [3]. The problem of deciding whether any finite number of terms are $\mathcal{V}$-dependent is called the *dependence-problem for* $\mathcal{V}$.

Let $\Gamma, \Delta \subseteq \mathrm{Eq}(\bar{y})$. We write $\Gamma \mathrel{|\!\!\sim}_{\mathcal{V}} \Delta$ if for any substitution $\sigma\colon \mathrm{Tm}(\bar{y}) \to \mathrm{Tm}(\omega)$ extended to equations by setting $\sigma(s \approx t) = \sigma(s) \approx \sigma(t)$,

$$\mathcal{V} \vDash \sigma(\Gamma) \quad \Longrightarrow \quad \mathcal{V} \vDash \sigma(\delta) \text{ for some } \delta \in \Delta.$$

Then we say that a set $\Delta \subseteq \mathrm{Eq}(\bar{y})$ is $\mathcal{V}$-*refuting* for $\bar{y}$ if for any equation $\varepsilon(\bar{y})$,

$$\mathcal{V} \nvDash \varepsilon \quad \Longleftrightarrow \quad \{\varepsilon\} \mathrel{|\!\!\sim}_{\mathcal{V}} \Delta.$$

Lemma 1. *For any $\mathcal{V}$-refuting set $\Delta(\bar{y})$ for $\bar{y} = \{y_1, \dots, y_n\}$, the terms $t_1, \dots, t_n \in Tm(\bar{x})$ are $\mathcal{V}$-dependent if and only if $\mathcal{V} \vDash \delta(t_1, \dots, t_n)$ for some $\delta \in \Delta$.*

Thus, for varieties that have a decidable equational theory and for which a finite $\mathcal{V}$-refuting set for any finite $\bar{y}$ can be constructed, the dependence-problem is decidable.

Example 2. We define $[n] \coloneqq \{1, \dots, n\}$. Let us consider the variety $\mathcal{L}at$ of all lattices and let

$$\Delta_n \coloneqq \Big\{ y_i \leq \bigvee_{j \in [n] \setminus \{i\}} y_j \mid i \in [n] \Big\} \cup \Big\{ \bigwedge_{j \in [n] \setminus \{i\}} y_j \leq y_i \mid i \in [n] \Big\}.$$

We can show that Δ_n is a $\mathcal{V}$-refuting set for $\{y_1, \dots, y_n\}$ and thus, the dependence-problem for $\mathcal{L}at$ is decidable.

Let us now turn our attention to modal semilattices studied for example in [2]. First we consider $\mathcal{MJS}$, the variety of $\langle A, \vee, \Box \rangle$-algebras such that $\langle A, \vee \rangle$ is a semilattice and

$$\Box a \vee \Box b \leq \Box(a \vee b), \quad \text{for all } a, b \in A.$$

The approach used for $\mathcal{L}at$ does not quite work for $\mathcal{MJS}$, since the modal depth of a formula can be arbitrarily large, but a more general approach works.

Lemma 3. *The following set of $\mathcal{MJS}$-inequations in $\bar{y}$ is $\mathcal{MJS}$-refuting for $\bar{y}$:*

$$\Delta_{\bar{y}} \coloneqq \{y \leq s \mid y \in \bar{y} \text{ and } s \neq s_1 \vee y \vee s_2\} \cup \{\Box^k y \leq y' \mid y, y' \in \bar{y} \text{ and } k \geq 0\}.$$

Note that $\Delta_{\bar{y}}$ is an infinite set of non-valid inequations for any $\bar{y} \neq \varnothing$. Let $\mathrm{md}(t)$ denote the modal depth of the term t and define $\mathrm{md}(s \leq t) = \max\{\mathrm{md}(s), \mathrm{md}(t)\}$ for the inequation $s \leq t$.

Theorem 4. *Let $t_1, \ldots, t_n \in Tm(\bar{x})$ and let $\bar{y} = \{y_1, \ldots, y_n\}$. Then $t_1, \ldots, t_n$ are $\mathcal{MJS}$-dependent if and only if there is an inequation $\delta \in \Delta^d_{\bar{y}}$ such that*

$$\mathcal{MJS} \vDash \delta(t_1, \ldots, t_n),$$

where $\Delta^d_{\bar{y}} \coloneqq \{\delta \in \Delta_{\bar{y}} \mid md(\delta) \leq d\}$ and $d \coloneqq \max\{md(t_1), \ldots, md(t_n)\}$.

Corollary 5. *The dependence-problem for $\mathcal{MJS}$ is decidable.*

Furthermore, we consider $\mathcal{MMS}$, the variety of $\langle A, \wedge, \Box\rangle$-algebras such that $\langle A, \wedge\rangle$ is a semilattice and

$$\Box a \wedge \Box b = \Box(a \wedge b), \quad \text{for all } a, b \in A.$$

Again, the approach used for $\mathcal{L}at$ does not work for $\mathcal{MMS}$, but studying free $\mathcal{MMS}$-algebras yields a simple method for deciding the dependence-problem for $\mathcal{MMS}$. The free $\mathcal{MMS}$-algebra over $m > 0$ generators is isomorphic to the following $\mathcal{MMS}$-algebra:

$$\langle (\mathcal{P}_{fin}(\mathbb{N}))^m \backslash \{\langle \varnothing, \ldots, \varnothing\rangle\}, \cup, \Box\rangle,$$

where $\mathcal{P}_{fin}(\mathbb{N})$ is the set of all finite subsets of $\mathbb{N}$, and $\cup, \Box$ are defined component-wise with $\Box\{a_1, \ldots, a_k\} \coloneqq \{a_1+1, \ldots, a_k+1\}$ for $a_1, \ldots, a_k \in \mathbb{N}$. Let $\mathbf{F}(\bar{x})$ denote the free $\mathcal{MMS}$-algebra over $\bar{x}$ and let the elements of $\mathbf{F}(\bar{x})$ be of the form $[t]$ for terms $t \in \mathrm{Tm}(\bar{x})$.

Theorem 6. *Let us consider the $\mathcal{MMS}$-terms $t_1, \ldots, t_n \in Tm(\bar{x})$. The following are equivalent:*

1. *$t_1, \ldots, t_n$ are $\mathcal{MMS}$-dependent.*
2. *There is $i \in \{1, \ldots, n\}$ such that for each variable x occurring in t_i one of the following holds:*
   - *(a) $[t_i] = [\Box^k x \wedge \Box^l x \wedge t'_i] \in \mathbf{F}(\bar{x})$, where $k \neq l$.*
   - *(b) There is $j \in \{1, \ldots, n\} \backslash \{i\}$ such that x also occurs in t_j.*

Corollary 7. *The dependence-problem for $\mathcal{MMS}$ is decidable.*

References

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