Some Topological Considerations on Orthogonality

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Introduction. This text explores some open lines of investigation related to a class of birelational structures called *orthogonal frames* [2, 1, 4]; these are relational structures $(X, \equiv_1, \equiv_2)$ provided with two equivalence relations which are *orthogonal* to each other, in the sense that $\equiv_1 \cap \equiv_2 = Id_X$.

The logic of orthogonal frames is the fusion $S5 \oplus S5$ [4].

Orthogonal frames are rather ubiquitous in the Modal Logic literature; among other things, they generalise products of Kripke frames [5]. This abstract delves deeper into a off-hand remark made in [4]: namely, the fact that both *subset spaces* and *topological spaces* can be 'seen as' orthogonal frames (in the sense that they are categorically equivalent to a certain class of these frames).

Definition 1. Recall that a *subset space* (X, τ) consists of a nonempty set X and a nonempty collection τ of subsets of X. A *topological space* is a subset space where $\varnothing, X \in \tau$, and τ is closed under arbitrary unions and finite intersections.

An orthogonal frame from a topological space.

Definition 2. Given a subset (or topological) space (X, τ), we construct its *associated orthogonal frame* $(\mathcal{O}, \equiv, \sim)$ as follows:
$\mathcal{O} = \{(x, U) : x \in U \in \tau\}$;
$(x, U) \equiv (y, V)$ iff $x = y$;
$(x, U) \sim (y, V)$ iff $U = V$.

The reader may check that the above frame is orthogonal, for if $(x, U) \equiv (y, V)$ and $(x, U) \sim (y, V)$, then $(x, U) = (y, V)$. A semantics for subset spaces is discussed in [3], where sentences in a bimodal language containing operators $\Box$ and K are evaluated with respect to pairs (x, U) such that $x \in U \in \tau$, as follows: $x, U \models \Box\phi$ iff $x, V \models \phi$ for all $V \in \tau \cap \mathcal{P}(U)$ with $x \in V$; $x, U \models K\phi$ iff $y, U \models \phi$ for all $y \in U$. If we define a partial order $\geq$ on the above frame $\mathcal{O}$ as follows:

$$(x, U) \geq (y, V) \text{ iff } x = y \text{ and } U \supseteq V$$
$$(\text{iff } (x, U) \equiv (y, V) \text{ and } (x, U)(\sim \circ \equiv)(z, W) \text{ for all } (z, W) \sim (y, V)),$$

then the usual relational semantics on the frame $(\mathcal{O}, \geq, \sim)$ coincides with the semantics outlined above.

Given that each subset or topological space has an orthogonal frame associated to it, characterising the exact class of such frames which are 'associated' to one of these spaces becomes the next natural question.

A topological space from an orthogonal frame.

Definition 3. An *orthogonal subset frame* is a frame $(\mathcal{O}, \equiv, \sim)$ where $\equiv$ and $\sim$ are equivalence relations satisfying:

(1) $\equiv \cap \sim = Id_{\mathcal{O}}$;

(2) if $a'(\equiv \circ \sim)b'$ and $b'(\equiv \circ \sim)a'$ for all $a' \sim a$ and for all $b' \sim b$, then $a \sim b$.

An *orthogonal topological frame* is an orthogonal subset frame which moreover satisfies:

(3) if $a \equiv b$, then there exists some c such that, for all $c' \sim c$, $c'(\equiv \circ \sim)a$ and $c'(\equiv \circ \sim)b$;

(4) for all nonempty $A \subseteq \mathcal{O}$, closed under $\sim$, there is some b such that

(4.1) $\forall a \in A$: $a(\equiv \circ \sim)b$; (4.2) $\forall b' \sim b\ \exists a' \in A$: $a' \equiv b'$.

The following holds:

Proposition 4. *An orthogonal subset (resp. topological) frame is isomorphic to the associated orthogonal frame of some unique-up-to-isomorphism subset (resp. topological) space.*

The corresponding space is $(X_{\mathcal{O}}, \tau_O)$, where $X_{\mathcal{O}}$ is the quotient set $\mathcal{O}/_{\equiv}$, and $\tau_\pi = \{\varnothing\} \cup \{U_\pi : \pi \in \mathcal{O}/_{\sim}\}$, where we define $U_\pi := \{\sigma \in X_{\mathcal{O}} : \sigma \cap \pi \neq \varnothing\}$. We note that, by orthogonality, $\sigma \in U_\pi$ if and only if $\sigma \cap \pi$ is a singleton, which provides us a natural way to construct the isomorphism alluded to in Prop. 4.

Theorem 5. *The category of orthogonal subset (resp. topological) frames is equivalent to the category of subset (resp. topological) spaces.*

Relation to point-free topology. In the *point-free topology* literature (e.g. [6]), a *frame* is a complete lattice $(L, \leq)$ such that, for all $A \subseteq L$ and $b \in X$, $(\bigvee A) \wedge b = \bigvee_{a \in A}(a \wedge b)$. In our orthogonal topological frames, the quotient set $\mathcal{O}/_{\sim}$ constitutes such a lattice (minus its minimum) along with the partial order: $[a]_\sim \preceq [b]_\sim$ iff $a(\leq \circ \sim)b$.

References

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