An approach à la de Vries to compact Hausdorff spaces and closed relations

M. Abbadini[1,*], G. Bezhanishvili[2], and L. Carai[1]

[1] University of Salerno, SA, Italy.
{mabbadini, lcarai}@unisa.it

[2] New Mexico State University, Las Cruces, New Mexico, U.S.A.
guram@nmsu.edu

De Vries [3] obtained a duality for the category KHaus of compact Hausdorff spaces and continuous functions. The objects of the dual category DeV are complete boolean algebras equipped with a proximity relation, known as de Vries algebras, and the morphisms are functions satisfying certain conditions. One drawback of DeV is that composition of morphisms is not usual function composition. We propose an alternative approach, where morphisms between de Vries algebras are certain relations and composition of morphisms is usual relation composition.

For our purpose, it is more natural to start with the category $\mathsf{KHaus}^{\mathsf{R}}$ whose objects are compact Hausdorff spaces and whose morphisms are closed relations (i.e., relations $R\colon X \to Y$ where R is a closed subset of $X \times Y$). This category was studied in [2], and earlier in [6] in the more general setting of stably compact spaces. The latter paper establishes a duality for $\mathsf{KHaus}^{\mathsf{R}}$ that generalizes Isbell duality [5] between KHaus and the category of compact regular frames and frame homomorphisms. This is obtained by generalizing the notion of a frame homomorphism to that of a preframe homomorphism. However, a similar duality in the language of de Vries algebras remained problematic (see [2, Rem. 3.14]). We resolve this problem as follows.

As in de Vries duality, with each compact Hausdorff space X we associate the de Vries algebra $(\mathsf{RO}(X), \prec)$, where $\mathsf{RO}(X)$ is the complete boolean algebra of regular open subsets of X and $\prec$ is defined on $\mathsf{RO}(X)$ by $U \prec V$ iff $\mathsf{cl}(U) \subseteq V$. The key is to associate with each closed relation $R\colon X \to Y$ the relation $S_R\colon \mathsf{RO}(X) \to \mathsf{RO}(Y)$ given by

$$U \, S_R \, V \text{ iff } R[\mathsf{cl}(U)] \subseteq V$$

($R[-]$ denotes the direct image under R). This defines a covariant functor from $\mathsf{KHaus}^{\mathsf{R}}$ to the category $\mathsf{DeV}^{\mathsf{S}}$ of de Vries algebras and compatible subordination relations between them (i.e., subordination relations $S\colon A \to B$ satisfying $S \circ \prec_A = S = \prec_B \circ S$). Our main result states that this functor is an equivalence. We then prove that this equivalence further restricts to an equivalence between KHaus and the wide subcategory $\mathsf{DeV}^{\mathsf{F}}$ of $\mathsf{DeV}^{\mathsf{S}}$ whose morphisms satisfy additional conditions. This yields an alternative to de Vries duality. The main advantage of $\mathsf{DeV}^{\mathsf{F}}$ is that composition of morphisms is usual relation composition.

While our main result establishes that $\mathsf{KHaus}^{\mathsf{R}}$ is equivalent to $\mathsf{DeV}^{\mathsf{S}}$, the choice of direction of morphisms is ultimately a matter of taste since morphisms are relations. In fact, both $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{DeV}^{\mathsf{S}}$ are dagger (and thus self-dual) categories, and hence our results could alternatively be stated in the language of duality rather than equivalence. (To obtain a duality, one associates to a closed relation $R\colon X \to Y$ the compatible subordination relation $T_R\colon \mathsf{RO}(Y) \to \mathsf{RO}(X)$ defined by $V \; T_R \; U$ iff $R^{-1}[\mathsf{cl}(V)] \subseteq U$. This assignment exhibits a duality between $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{DeV}^{\mathsf{S}}$, which restricts to a duality between KHaus and a wide subcategory of $\mathsf{DeV}^{\mathsf{S}}$ whose morphisms satisfy conditions that are "dual" to those satisfied by the morphisms of $\mathsf{DeV}^{\mathsf{F}}$.)

[*] Speaker.

Our proof builds on a generalization of Halmos duality, which in turn generalizes Stone duality. By Stone duality, the category of Stone spaces (i.e. zero-dimensional compact Hausdorff spaces) and continuous maps is dually equivalent to the category of boolean algebras and boolean homomorphisms. Halmos [4] generalized this result to a duality between the category of Stone spaces and continuous relations and the category of boolean algebras and functions preserving finite joins. This approach can be further generalized by working with closed relations instead of continuous ones. As shown in [1], a closed binary relation on a Stone space X corresponds to a subordination relation on the boolean algebra $\mathsf{Clop}(X)$ of clopen subsets of X. The notion of a subordination relation on a boolean algebra generalizes to that of a subordination relation between two boolean algebras. This yields the category BA^{S} of boolean algebras and subordination relations between them (identity is $\leq$ and composition is relation composition). There is an equivalence (and also a dual equivalence) of categories between BA^{S} and the full subcategory $\mathsf{Stone}^{\mathsf{R}}$ of $\mathsf{KHaus}^{\mathsf{R}}$ consisting of Stone spaces. This result generalizes Stone and Halmos dualities. For a more general result in the context of Priestley spaces and bounded distributive lattices we refer to [7]. Using Karoubi envelopes and Gleason covers, we derive our equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and $\mathsf{DeV}^{\mathsf{S}}$ from the equivalence between $\mathsf{Stone}^{\mathsf{R}}$ and BA^{S}.

In [8, Thm. 26], the category of stably compact spaces and continuous functions was shown to be equivalent to the category of strong proximity lattices and approximable relations (see also [10] for a proof that explicitly uses Karoubi envelopes). This equivalence was specialized to compact Hausdorff spaces in [9]. In this case, to a compact Hausdorff space X is associated the set $\{(U, K) \mid U$ open subset of X, K closed subset of X, $U \subseteq K\}$, equipped with an appropriate structure. The difference between this assignment and ours (based on regular open sets) makes the axioms of the structures in [9] incomparable to ours.

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