Torsion theories and coverings of preordered groups

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A *preordered group* $(G, \leq)$ is a (not necessarily abelian) group $G = (G, +, 0)$ endowed with a preorder (i.e. a relation that is both reflexive and transitive) $\leq$ which is compatible with the addition $+$ of the group G: for any a, b, c, d in G, $a \leq c$ and $b \leq d$ implies that $a + b \leq c + d$. Given two preordered groups $(G, \leq_G)$ and $(H, \leq_H)$, a morphism f from $(G, \leq_G)$ to $(H, \leq_H)$ is said to be a *morphism of preordered groups* when $f\colon G \to H$ is a preorder preserving group morphism. All preordered groups and morphisms between them form a category, the category $\mathsf{PreOrdGrp}$ of preordered groups. The first categorical properties of $\mathsf{PreOrdGrp}$ have been studied in [4] by Clementino, Martins-Ferreira and Montoli. Among other things, they recall that the category $\mathsf{PreOrdGrp}$ of preordered groups is isomorphic to the category whose objects are pairs (G, M), where G is a group and M a submonoid of G closed under conjugation in G (that is $g + m - g \in M$ for any $g \in G$ and $m \in M$), and whose arrows $f\colon (G, M) \to (H, N)$ are group morphisms $f\colon G \to H$ satisfying the condition $f(M) \subseteq N$. The submonoid M in a given preordered group (G, M) is called the *positive cone* of G and is usually written P_G. Two important results of the article [4] are the fact that $\mathsf{PreOrdGrp}$ is a *normal category* [11] and that the *effective descent morphisms* in this context exactly coincide with the normal epimorphisms.

In this talk, we first present a *torsion theory* [1, 3] in the category $\mathsf{PreOrdGrp}$. This is given by the pair $(\mathsf{Grp}, \mathsf{ParOrdGrp})$ where Grp and $\mathsf{ParOrdGrp}$ are two full and replete subcategories of $\mathsf{PreOrdGrp}$ described as follows. The objects of Grp are preordered groups of the form (G, G), i.e. the preordered groups whose positive cone is the entire group. Via the above mentioned isomorphism of categories, they correspond to groups G endowed with the *indiscrete relation*: $a \leq b$ for any pair of elements a and b of G. The objects of $\mathsf{ParOrdGrp}$ are for their part given by *partially ordered groups*, in other words by preordered groups whose preorder is antisymmetric. Alternatively, they can also be seen as pairs (G, P_G) where the positive cone P_G is a *reduced monoid* (in the sense that the only element in P_G having its inverse also in P_G is the neutral element 0).

From this torsion theory, we directly get (thanks to the unique Proposition in [10]) the following result: $\mathsf{ParOrdGrp}$ is a *(normal epi)-reflective* subcategory, while Grp is *(normal mono)-coreflective* in $\mathsf{PreOrdGrp}$. In particular, the functor $F\colon \mathsf{PreOrdGrp} \to \mathsf{ParOrdGrp}$, associating, to any preordered group (G, P_G), the partially ordered group $(G/N_G, P_G/N_G)$ (where N_G is the normal subgroup of elements x in G such that both x and $-x$ are in the positive cone P_G), is a *reflector*. We can prove that it has moreover *stable units* [2], hence it naturally induces a factorization system $(\mathcal{E}, \mathcal{M})$, where $\mathcal{E}$ is the class of morphisms in $\mathsf{PreOrdGrp}$ inverted by the functor F and $\mathcal{M}$ the class of *trivial coverings* [2] of $\mathsf{PreOrdGrp}$. A fairly simple characterization has been obtained for this last class: a morphism $f\colon (G, P_G) \to (H, P_H)$ in $\mathsf{PreOrdGrp}$ is a trivial covering (i.e. is in $\mathcal{M}$) if and only if its restriction $\phi\colon N_G \to N_H$ to N_G is an isomorphism of

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groups. In order to be able to describe all *coverings*, we then need to prove two intermediate results. In the proof of one of them, we build, for any preordered group (G, P_G), a partially ordered group (H, P_H) as well as an effective descent morphism $\pi_2 \colon (H, P_H) \to (G, P_G)$ from (H, P_H) to (G, P_G). Thanks to these two intermediate results, it is next possible to apply a theorem by Everaert and Gran [5] and then to get a description of coverings in the category $\mathsf{PreOrdGrp}$ of preordered groups. These are given by those morphisms in $\mathsf{PreOrdGrp}$ whose kernel is a partially ordered group. Furthermore, we also deduce that the factorization system $(\mathcal{E}', \mathcal{M}^*)$ (where $\mathcal{E}'$ is the "*stabilization*" of the class $\mathcal{E}$ and $\mathcal{M}^*$ the "*localization*" of the class $\mathcal{M}$) is *monotone-light*.

We then notice that the coverings in $\mathsf{PreOrdGrp}$ can be classified in terms of *internal actions* of the *Galois groupoid* associated with the above mentioned effective descent morphism π_2. Note that, besides its interest for the study of coverings, this last result also provides a new example of application of a theorem by Janelidze, Márki and Tholen [9] in a *non-exact* setting. Finally, we observe that, in addition to the torsion theory already exposed previously, there is also a *pretorsion theory* [6, 7] in $\mathsf{PreOrdGrp}$. The torsion-free subcategory is the same as for the torsion theory (i.e. $\mathsf{ParOrdGrp}$) while the torsion subcategory, denoted by $\mathsf{ProtoPreOrdGrp}$, is given by the full subcategory of $\mathsf{PreOrdGrp}$ whose objects are preordered groups (G, P_G) for which the positive cone P_G is a group. As shown in [4], the objects of $\mathsf{ProtoPreOrdGrp}$ are the so-called *protomodular objects* of $\mathsf{PreOrdGrp}$. By using the notation with the preorders, it is easily seen that these objects are actually preordered groups endowed with an equivalence relation, i.e. they are internal groups in the category of preordered sets.

All these results can be found in the article [8] written in collaboration with Marino Gran. Note that these have recently been extended to the broader context of *V-groups* in [12] (for a suitable *quantale* V).

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