Admissibility of Π_2-Inference Rules: interpolation, model completion, and contact algebras

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The use of non-standard inference rules has a long tradition in modal logic starting from the pioneering work of Gabbay [4]. We consider a class of non-standard rules called Π_2-rules. An inference rule ρ is a Π_2-*rule* if it is of the form

$$\frac{F(\underline{\varphi}/\underline{x},\underline{y}) \to \chi}{G(\underline{\varphi}/\underline{x}) \to \chi}$$

where $F(\underline{x},\underline{y}), G(\underline{x})$ are propositional formulas. We say that θ is obtained from ψ by an application of the rule ρ if $\psi = F(\underline{\varphi}/\underline{x},\underline{y}) \to \chi$ and $\theta = G(\underline{\varphi}/\underline{x}) \to \chi$, where $\underline{\varphi}$ is a tuple of formulas, χ is a formula, and $\underline{y}$ is a tuple of propositional letters not occurring in $\underline{\varphi}, \chi$.

Rather little is known about the problem of recognizing *admissibility* for Π_2-rules. We show that there are tools already available in the literature on modal logic that can be fruitfully employed for studying admissibility of Π_2-rules. We present three different strategies for recognizing admissibility over a propositional modal system $\mathcal{S}$. In the following we will assume that ρ is given by the formulas F and G as above.

Conservative extensions, uniform interpolation, and model completions

Our first strategy applies to modal systems with the interpolation property. We determine admissibility of Π_2-rules via *conservative extensions*. We say that $\varphi(\underline{x}) \wedge \psi(\underline{x},\underline{y})$ is a conservative extension of $\varphi(\underline{x})$ in $\mathcal{S}$ if for every formula $\chi(\underline{x})$, we have that $\vdash_{\mathcal{S}} \varphi(\underline{x}) \wedge \psi(\underline{x},\underline{y}) \to \chi(\underline{x})$ implies $\vdash_{\mathcal{S}} \varphi(\underline{x}) \to \chi(\underline{x})$.

Theorem 1. *Assume that $\mathcal{S}$ has the interpolation property. A Π_2-rule ρ is admissible in $\mathcal{S}$ iff $G(\underline{x}) \wedge F(\underline{x},\underline{y})$ is a conservative extension of $G(\underline{x})$ in $\mathcal{S}$. In addition, if conservativity is decidable in $\mathcal{S}$, then Π_2-rules are effectively recognizable in $\mathcal{S}$.*

Our second strategy allows to determine the admissibility of Π_2-rules in systems with a universal modality. We use *uniform interpolation* which is a strengthening of ordinary interpolation. If $\varphi(\underline{x},\underline{y})$ is a formula, its right global uniform pre-interpolant $\forall_{\underline{x}}\varphi(\underline{y})$ is a formula such that for every $\psi(\underline{y},\underline{z})$ we have that

$$\psi(\underline{y},\underline{z}) \vdash_{\mathcal{S}} \varphi(\underline{x},\underline{y}) \quad \text{iff} \quad \psi(\underline{y},\underline{z}) \vdash_{\mathcal{S}} \forall_{\underline{x}}\varphi(\underline{y}).$$

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Theorem 2. *Suppose that $\mathcal{S}$ has uniform global pre-interpolants and a universal modality $[\forall]$. Then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff*

$$\vdash_{\mathcal{S}} [\forall]\forall_{\underline{y}}(F(\underline{x},\underline{y}) \to z) \to (G(\underline{x}) \to z).$$

Moreover, if $\mathcal{S}$ is decidable and global uniform interpolants are computable in $\mathcal{S}$, then Π_2-rules are effectively recognizable in $\mathcal{S}$.

Our third strategy exploits the connection between Π_2-rules and model-theoretic machinery. With each Π_2-rule ρ, we associate the following $\forall\exists$-statement in the *first-order* language of $\mathcal{S}$-algebras:

$$\Pi(\rho) := \forall \underline{x}, z\Big(G(\underline{x}) \nleq z \Rightarrow \exists \underline{y} : F(\underline{x},\underline{y}) \nleq z\Big).$$

Theorem 3. *Suppose that $\mathcal{S}$ has a universal modality and let $T_{\mathcal{S}}$ be the theory of the simple non-degenerate $\mathcal{S}$-algebras. If $T_{\mathcal{S}}$ has a model completion $T^{\star}_{\mathcal{S}}$, then a Π_2-rule ρ is admissible in $\mathcal{S}$ iff $T^{\star}_{\mathcal{S}} \models \Pi(\rho)$.*

As a consequence, we obtain an alternative way to recognize admissibility.

Corollary 4. *Let $\mathcal{S}$ be a system with universal modality that is decidable and locally tabular. If simple $\mathcal{S}$-algebras enjoy the amalgamation property, then admissibility of Π_2-rules in $\mathcal{S}$ is effective.*

Contact algebras and admissibility in $\mathsf{S^2IC}$.

Recently, there has been a renewed interest in non-standard rules in the context of the region-based theories of space. One of the key algebraic structures in these theories is that of *contact algebras*. Compingent algebras are contact algebras satisfying two $\forall\exists$-sentences (aka Π_2-sentences). De Vries [3] established a duality between complete compingent algebras and compact Hausdorff spaces. This duality led to new logical calculi for compact Hausdorff spaces in [1, 2]. Key to these approaches is a development of logical calculi corresponding to contact algebras. In [2] such a calculus is called the *strict symmetric implication calculus* and is denoted by $\mathsf{S^2IC}$. The extra Π_2-axioms of compingent algebras then correspond to non-standard Π_2-rules, which turn out to be admissible in $\mathsf{S^2IC}$.

We apply our third strategy to study admissibility of Π_2-rules in $\mathsf{S^2IC}$. We also show that the admissibility problem for $\mathsf{S^2IC}$ is co-NExpTime-complete. This is done by using the model completion of the theory of contact algebras. Moreover, we explicitly list three sentences that, together with the axioms of contact algebras, axiomatize the model completion.

Theorem 5. *The model completion of the theory of contact algebras is finitely axiomatizable*

References

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