Frobenius structures in autonomous categories

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In a series of successive works the following result is established:

Theorem 1 (See [7, 4, 5, 10, 9])**.** *The quantale $[L, L]$ of sup-preserving endomaps of a complete lattice L is a Frobenius quantale if and only if L is a completely distributive lattice.*

We give a proof of this result that relies on the $*$-autonomous structure of SLatt, the category of complete lattices and sup-preserving maps. In doing so, we generalise this result to arbitrary $*$-autonomous categories. Recall that an object A of an autonomous category $\mathcal{V} = (V, I, \otimes, \alpha, \lambda, \rho, [-,-], ev)$ is *nuclear* if the canonical map $\texttt{mix} : A^* \otimes A \to [A, A]$ is an isomorphism, where A^*, the dual of A, is the internal hom $[A, I]$. We rely on the following characterization of completely distributive lattices:

Theorem 2 (See [8, 6])**.** *Nuclear objects in* SLatt *are exactly the completely distributive lattices.*

A main tool that we use is the notion of dual pair:

Definition 3. A dual pair in a monoidal category $\mathcal{V}$ is a triple (A, B, ϵ), with A, B objects of $\mathcal{V}$ and $\epsilon : A \otimes B \longrightarrow I$, yielding via Yoneda natural isomorphisms

$$\hom(X, B) \simeq \hom(A \otimes X, I) \text{ and } \hom(X, A) \simeq \hom(X \otimes B, I)\,.$$

We informally say that (A, B) is a dual pair, leaving aside the arrow ϵ. Clearly, (A, A^*) is a dual pair in any $*$-autonomous category. This notion provides the framework by which to study objects that are dual to each other only up to isomorphism: for example $(A^* \otimes A, [A, A])$ is a dual pair in any $*$-autonomous category and, for any complete lattice L, (L, L^{op}) is a dual pair in SLatt. Some elementary properties of dual pairs are immediate, for instance, if (A, B) is a dual pair, then A and B are both reflexive, that is, isomorphic to their double dual.
If (A, B) is a dual pair and A is a semigroup, then A acts on B on the left and on the right. The left and right actions, written here α^ℓ and α^ρ, correspond, in the category SLatt, to the two implications of a quantale, see e.g. [7, 3]. We define then generalized Frobenius quantales in arbitrary autonomous categories as follows:

Definition 4. A *Frobenius structure* is a tuple (A, B, μ_A, l, r) where (A, B) is a dual pair, (A, μ_A) is a semigroup, l and r are adjoint invertible maps from A to B such that the diagram below on the left commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{A \otimes r} & A \otimes B \\ {\scriptstyle l \otimes A}\downarrow & & \downarrow{\scriptstyle \alpha^\ell_A} \\ B \otimes A & \xrightarrow[\alpha^\rho_A]{} & B \end{array} \qquad\qquad \begin{array}{ccc} B \otimes B & \xrightarrow{B \otimes l^{-1}} & B \otimes A \\ {\scriptstyle r^{-1} \otimes B}\downarrow & & \downarrow{\scriptstyle \alpha^\rho} \\ A \otimes B & \xrightarrow[\alpha^\ell]{} & B. \end{array} \tag{1}$$

By saying that l and r are adjoint, we mean that their transposes differ by a symmetry: $\epsilon \circ (A \otimes l) = \epsilon \circ (A \otimes r) \circ \sigma_{A,A}$. Definitions of Frobenius structures in various kind of monoidal

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categories already exist in the literature [1, 11, 3, 4]. In these works the units (and co-units) play an important role. Following [2], where we have argued that neither dualizing elements nor units are needed in order to define Frobenius quantales, Definition 4 does not require units. When (A, μ_A) is a semigroup in SLatt, that is, a quantale, and $B = A^{op}$, the arrows l and r play the role of the two negations in a Frobenius quantales, they are adjoint in that they form a Galois connection. The diagram on the left of (1) may be understood as the equation $y\backslash^{\perp}x = y^{\perp}/x$ linking negations and implications.
Couples of quantales, as defined in [4], are apparently the most similar to the Frobenius structures defined here. For a couple of quantales, however, only one negation (not necessary invertible) is considered and the right diagram of (1) is required to be commutative; once more, for the negation to be classical, one requires the existence of a dualizing element and thus of a unit. Definition 4 does not mention dualizing elements and implies the commutativity of the right diagram of (1). If we let μ_B be the diagonal of this diagram, then we have:

Lemma 5. *If (A, B, μ_A, l, r) is a Frobenius structure, then so is $(B, A, \mu_B, r^{-1}, l^{-1})$.*

It is now immediate to derive the following:

Theorem 6. *If A is nuclear, then there is a map l such that $([A, A], \circ, [A, A]^*, l, l)$ is a Frobenius structure.*

Indeed, $A^* \otimes A$ is canonically a semigroup and if the canonical map $\mathtt{mix} : A^* \otimes A \longrightarrow [A, A]$ is invertible, then $(A^* \otimes A, [A, A], \mu_{A^*\otimes A}, \mathtt{mix}, \mathtt{mix})$ is a Frobenius structure. We derive the theorem, since Frobenius structures are closed up to isomorphism and using Lemma 5. Theorem 6 is actually an instance of Corollary 3.3 in [11]. However, it can be further generalised: if $\mathtt{mix}$ is not invertible but the underlying $*$-autonomous category has some nice factorisation system, then the image of $\mathtt{mix}$ is the support of a Frobenius structure. This is a consequence of the following statement:

Theorem 7. *Let $\mathcal{V}$ be a $*$-autonomous category with a factorization system. Let (A, μ_A) be a semigroup and (A, B) be a dual pair. Let $f : A \longrightarrow B$ be a map, put $\psi_A = \epsilon \circ (A \otimes f)$ and suppose that $\psi_A = \psi_A \circ \sigma_{A,A}$. Factor f as $f = m \circ e$ with $e : A \longrightarrow C$ epi and $m : C \longrightarrow B$ mono. If C is a magma with multiplication μ_C and e is a magma homomorphism, then there exist maps $\psi_C : C \otimes C \longrightarrow I$ and $g : C \longrightarrow C^*$, transposing into each other, making (C, C^*, μ_C, g, g) into a Frobenius structure.*

The converse of Theorem 6 actually holds if we add another condition.

Definition 8. An objet A of a monoidal category is *pseudo-affine* if the tensor unit I embeds into A as a retract. A monoidal category is *pseudo-affine* if every object which is not terminal nor initial is affine.

For example, the category SLatt is pseudo-affine.

Theorem 9. *In a $*$-autonomous category, if A is a pseudo-affine object and the canonical monoid $([A, A], \circ)$ is part of a Frobenius structure, then A is nuclear.*

The proof of this theorem relies on the following ideas. If A, B are pseudo-affine, then the following statement holds:

Lemma 10. *If $A \otimes f = g \otimes B : A \otimes X \otimes B \longrightarrow A \otimes Y \otimes B$, then there exists $h : X \longrightarrow Y$ such that $f = h \otimes B$ and $g = A \otimes h$.*

This lemma is applied to the dual pair $([A, A], A^* \otimes A)$ as follows: since in this case $\alpha^\rho = A^* \otimes ev_{A,A}$, the diagonal of the diagram on the right of (1) is of the form $A^* \otimes f$. Considering the explicit form of α^ℓ, we also deduce that this same diagonal is of the form $g \otimes A$. Since both A and A^* are pseudo-affine, we deduce, by the last lemma, the existence of a map $\epsilon : A \otimes A^* \longrightarrow I$. Since $A^* \otimes A$ is isomorphic as a semigroup to $[A, A]$, it is also unital, thereby there exists a map $\eta : I \longrightarrow A^* \otimes A$. Since A and A^* are pseudo-affine, tensoring with them is faifthul and we deduce from the monoid diagrams for $A^* \otimes A$ that (A, A^*, η, ϵ) is an adjunction. We therefore deduce Theorem 9 from the fact that nuclear objects in an autonomous category are exactly the adjoints, left or right, since the category is symmetric.

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