Lattice-ordered groups via distributive lattice-ordered monoids

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Abstract

We show that the equational theory of lattice ordered groups (ℓ-groups) reduces to that of distributive lattice-ordered monoids (DLMs) and that DLMs have the finite model property. Furthermore, we provide an axiomatization for the variety of representable DLMs and show that they satisfy fewer equations than the inverse-free reducts of representable ℓ-groups. We also provide a link to right orders on the free group and the free monoid.

A *lattice-ordered group* (*ℓ-group*) consists of a lattice and a group on the same set such that multiplication distributes over meet and join. Lattice-ordered groups have a long history and a rich algebraic theory. We denote by LG the variety of ℓ-groups.

A *distributive lattice-ordered monoid* (*DLM*) is a lattice and a monoid on the same set such that multiplication distributes over meet and join and lattice distributivity holds. We denote by DLM the variety of DLMs. It is easy to see that lattice distributivity follows from the ℓ-group axioms, so the lattice-monoid reducts of ℓ-groups are DLMs.

An *abelian* ℓ-group is an ℓ-group where multiplication is commutative; a commutative DLM is one where multiplication is commutative. We denote by ALG and CDLM the corresponding varieties.

Given an ℓ-group equation ε, by 'clearing denominators' ε can be transformed to an inverse-free equation ε' such that $\mathsf{ALG} \models \varepsilon \Leftrightarrow \mathsf{ALG} \models \varepsilon'$. Unfortunately, Repniskii [5] showed that the further equivalence $\mathsf{ALG} \models \varepsilon' \Leftrightarrow \mathsf{CDLM} \models \varepsilon'$ fails (there are inverse-free equations that hold in ALG but not in CDLM), so we cannot easily reduce the equational theory of ALG to that of CDLM. Actually, Repniskii provided an infinite axiomatization of the inverse-free equational theory of ALG relative to that of CDLM and proved no finite one exists.

We prove [1] that this discrepancy also holds in the representable case. An ℓ-group (DLM) is called *representable* (or *semilinear*) if it is a subdirect product of totally-ordered ℓ-groups (DLMs, respectively); the associated varieties are denoted by RLG and RDLM. In [1] we provide an equational basis for RDLM.

Theorem 1. *The equivalence* $\mathsf{RLG} \models \varepsilon \Leftrightarrow \mathsf{RDLM} \models \varepsilon$ *fails, for some inverse-free equation* ε.

It comes as a surprise that even though for both the commutative and the representable case more equations are satisfied by the corresponding inverse-free reducts of ℓ-groups than by the corresponding DLMs, in the general case this discrepancy does not appear.

Theorem 2. *The equivalence* $\mathsf{LG} \models \varepsilon \Leftrightarrow \mathsf{DLM} \models \varepsilon$ *holds, for every inverse-free equation* ε.

As part of the proof of Theorem 2, we prove that the variety DLM is generated by the DML of order permutations of the chain of the rationals.

Due to the lack of commutativity, clearing denominators is not as obvious in LG as in ALG, so it is not clear if we can make use of Theorem 2 in order to reduce the equational theory of LG to that of DLM by a string of equivalences:

$$\mathsf{LG} \models \varepsilon \Leftrightarrow \mathsf{LG} \models \varepsilon' \Leftrightarrow \mathsf{DLM} \models \varepsilon' \qquad (*)$$

where for every ℓ-group equation ε, ε' is an inverse-free equation corresponding to it. It was the second equivalence that failed in the commutative case and the first one that was true; now the second equivalence is true. It is equally surprising that we can also 'clear denominators' in the non-commutative case (this is not possible in groups, but we prove it holds in ℓ-groups by making crucial use the join operation). The following result was inspired by the proof of density-elimination in proof theory.

Theorem 3. *For every ℓ-group equation ε, there exists an (effectively constructible) inverse-free equation ε' such that $\mathsf{LG} \models \varepsilon \Leftrightarrow \mathsf{LG} \models \varepsilon'$.*

Therefore, (*) holds and thus we obtain the following result.

Corollary 4. *The equational theory of LG can be effectively reduced to the equational theory of DLM.*

There exists an analytic Gentzen-Dunn-Mints-style sequent calculus for DLM (so it enjoys cut elimination and has the subformula property), but unfortunately it is not known how to extract a decision procedure from it. However, the following result shows how to obtain decidability of the equational theory of DLM, and ultimately of LG by invoking (*). This yields an alternative proof of the decidability of LG (see [4], [2]) that avoids relying on Holland's embedding theorem [3].

Theorem 5. *The variety DLM has the finite model property.*

Finally, we connect our study to the theory of right orders (and even to right preorders). A *right order* on a monoid is a total order that is compatible with right multiplication.

Theorem 6. *Every right order on the free monoid over a set X extends to a right order over the free group over X.*

References

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