

From residuated lattices to ℓ -groups via free nuclear preimages

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Two fundamental constructions on partially ordered monoids, or *pomonoids*, are so-called *nuclear images* and *conuclear images*. A *nucleus* is closure operator γ on a pomonoid such that $\gamma x \cdot \gamma y \leq \gamma(x \cdot y)$, while a *conucleus* is an interior operator σ on a pomonoid such that $\sigma x \cdot \sigma y \leq \sigma(x \cdot y)$ and $\sigma e = e$, where e denotes the multiplicative unit. The image of a nucleus or a conucleus on a pomonoid (on a residuated lattice) can be equipped with the structure of a pomonoid (a residuated lattice), although not all of its operations will coincide with the corresponding operations in the original algebra.

Nuclear images allow us to construct many of the ordered algebras which arise in non-classical logic (such as pomonoids, semilattice-ordered monoids, or residuated lattices) from cancellative ones. Conuclear images then allow us to construct some of these cancellative algebras from partially ordered and lattice-ordered groups (pogroups and ℓ -groups). In this work, we consider the problem of which algebras arise as nuclear images of conuclear images of pogroups and ℓ -groups, and more generally as nuclear images of cancellative structures

In asking this question, we follow a line of research stemming from the classical result of Mundici [2] that MV-algebras are precisely the unit intervals of negative cones of Abelian ℓ -groups. That is, each MV-algebra can be constructed from an Abelian ℓ -group in two steps: first restricting to the negative cone of the ℓ -group (consisting the elements below e), and then further restricting to some interval $[u, e]$ in this negative cone, adjusting the operations of the ℓ -group accordingly at each step. These constructions are special cases of conuclear and nuclear images: take $\sigma x := e \wedge x$ and $\gamma x := x \vee u$. Mundici's result was later extended by Galatos & Tsinakis [1] to so-called GMV-algebras, which drop the requirements of commutativity, integrality, and boundedness. These algebras are precisely the nuclear images of cancellative GMV-algebras, which in turn are precisely the kernel images of ℓ -group, where a *kernel* is a conucleus whose image is downward closed. Galatos & Tsinakis prove this by extending Mundici's technique of good sequence to the setting of GMV-algebras. This technique, however, does not appear to apply outside the setting of GMV-algebras.

In order to extend these results beyond GMV-algebras, we first identify which pomonoids or *s ℓ -monoids* (join-semilattice-ordered monoids) are nuclear images of cancellative pomonoids or *s ℓ -monoids*, i.e. those pomonoids or *s ℓ -monoids* which satisfy

$$x \cdot y \leq x \cdot z \implies y \leq z, \quad x \cdot z \leq y \cdot z \implies x \leq y.$$

The key construction here is the *free nuclear preimage*. The nuclear image construction yields a functor from the category of *nuclear pomonoids* or *nuclear s ℓ -monoids* (pomonoids or *s ℓ -monoids* equipped with a nucleus) into the category of pomonoids or *s ℓ -monoids*. The free nuclear preimage is the left adjoint of this functor. We provide an explicit description of free nuclear preimages of pomonoids and *s ℓ -monoids* and use it to prove the following theorems, where a pomonoid is called *integrally closed* if it satisfies

$$x \cdot y \leq x \implies y \leq e, \quad x \cdot y \leq y \implies x \leq e.$$

Theorem 1. *The nuclear images of (integral) [commutative] cancellative pomonoids are precisely the integrally closed (integral) [commutative] pomonoids.*

The analogous theorem for commutative sl -monoids involves what we call the *square condition*, which is a certain infinite set of equations in the signature of sl -monoids.

Theorem 2. *The nuclear images of [distributive] cancellative integral sl -monoids are precisely the integral sl -monoids. The nuclear images of [distributive] commutative cancellative (integral) sl -monoids are precisely the commutative integrally closed (integral) sl -monoids satisfying the square condition.*

While the subpomonoids of Abelian pogroups are precisely the commutative cancellative pomonoids, the subpomonoids of general pogroups defy any simple description. However, using a proof-theoretic argument, we can nevertheless improve the above characterization of nuclear images of cancellative pomonoids to one of nuclear images of subpomonoids of pogroups.

Theorem 3. *The nuclear images of (integral) subpomonoids of pogroups are precisely the integrally closed (integral) pomonoids.*

A major task which remains to be done is to extend this proof-theoretic argument from pogroups to ℓ -groups, aiming to prove the conjecture that the nuclear images of (integral) sub- sl -monoids of ℓ -groups are precisely the integrally closed (integral) sl -monoids.

The free nuclear preimage construction also yields a syntactic characterization of which ordered quasivarieties of pomonoids (of sl -monoids), i.e. classes axiomatized by implications between a finite set of inequalities and a single inequality, are closed under nuclear images.

Theorem 4. *An ordered quasivariety of pomonoids (of sl -monoids) is closed under nuclear images if and only if it is axiomatized by a set of simple quasi-equations, i.e. ones where in each premise $t \leq u$ the term u is a variable.*

For example, the quasi-inequalities which define integrally closed pomonoids are simple, while the quasi-inequalities which define cancellative pomonoids are not.

Finally, returning to our original motivation, in the finite case we can extend these results about sl -monoids to results about residuated lattices.

Theorem 5. *The finite nuclear images of (commutative) cancellative [integral] residuated lattices are precisely the finite integral residuated lattices (satisfying the square condition).*

Theorem 6. *The finite nuclear images of conuclear images of Abelian ℓ -groups are precisely the finite integral residuated lattices satisfying the square condition.*

One might hope to extend this argument to arbitrary ℓ -groups, aiming to prove that the finite nuclear images of conuclear images of ℓ -groups are precisely the finite integral residuated lattices.

References

- [1] Nikolaos Galatos and Constantine Tsinakis. Generalized MV-algebras. *Journal of Algebra*, 283(1):254–291, 2005.
- [2] Daniele Mundici. Interpretation of AF C^* -algebras in Lukasiewicz sentential calculus. *Journal of Functional Analysis*, 65:15–63, 1986.