Projective unification through duality

Philippe Balbiani and Quentin Gougeon*

Institut de recherche en informatique de Toulouse
CNRS-INPT-UT3, Toulouse University, Toulouse, France
philippe.balbiani@irit.fr quentin.gougeon@irit.fr

In a propositional language, substitutions can be defined as functions mapping variables to formulas. For reasons related to Unification Theory [BS01, Section 2], it is usually considered that such functions are almost everywhere equal to the identity function. According to this point of view, which is the one usually considered within the context of modal logics [BR11, Dzi07, Ghi00], a substitution is a function $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ where $\mathcal{L}_P$ (resp. $\mathcal{L}_Q$) is the set of all formulas with variables in a finite set P (resp. Q), and satisfying (⧫) $\sigma(\circ(\varphi_1, \dots, \varphi_n)) = \circ(\sigma(\varphi_1), \dots, \sigma(\varphi_n))$ for all n-ary connectives $\circ$ of the language and all formulas $\varphi_1, \dots, \varphi_n \in \mathcal{L}_P$.

A formula $\varphi \in \mathcal{L}_P$ is **L**-unifiable if **L** contains instances of φ. In that case, any substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ such that $\sigma(\varphi) \in \mathbf{L}$ counts as a **L**-unifier of φ. A **L**-unifiable formula $\varphi \in \mathcal{L}$ is projective if it possesses a projective **L**-unifier, that is to say a **L**-unifier σ such that $\varphi \vdash_{\mathbf{L}} \sigma(p) \leftrightarrow p$ holds for all $p \in P$. Such unifiers are interesting because they constitute by themselves minimal complete sets of unifiers [BR11, Dzi07, Ghi00]. For this reason, it is of the utmost importance to be able to determine if a given formula is projective. Ghilardi's proof that transitive modal logics such as **K4** and **S4** are finitary is based on projective unifiers [Ghi00]. In [Sło12], Słomczyńska uses projective unifiers to determine the unification type of some implicational fragments of intuitionistic propositional logic.

Now, condition (⧫) may evoke homomorphism properties. Following this observation, Unification Theory was also formalized and studied in an algebraic setting [Ghi97, Sło12]. Indeed, let us consider the Lindenbaum algebra $\mathbf{A}_P$ obtained by taking the quotient of $\mathcal{L}_P$ modulo the relation $\equiv_{\mathbf{L}}$ of **L**-equivalence. One can associate to a substitution $\sigma : \mathcal{L}_P \to \mathcal{L}_Q$ the map $\sigma^{\triangleright} : \mathbf{A}_P \to \mathbf{A}_Q$ by setting $\sigma^{\triangleright}([\varphi]_{\mathbf{L}}) := [\sigma(\varphi)]_{\mathbf{L}}$ for any formula $\varphi \in \mathcal{L}_P$, whose equivalence class modulo $\equiv_{\mathbf{L}}$ is denoted by $[\varphi]_{\mathbf{L}}$. In this perspective, condition (⧫) then truly expresses the homomorphic character of $\sigma^{\triangleright}$. Obviously, this association between substitutions and homomorphisms of Lindenbaum algebras is one-to-one modulo $\simeq_{\mathbf{L}}$: substitutions associated to the same homomorphism are equivalent modulo $\simeq_{\mathbf{L}}$. Then properties of substitutions, such as being a **L**-unifier of a formula, admit an algebraic counterpart too.

In this work, we combine this correspondence with a more traditional one, provided by *Duality Theory*. For any set P of variables, there is indeed a tight connection between the Lindenbaum algebra $\mathbf{A}_P$ and the canonical frame $\mathfrak{F}_P$ of **L** over P, determined by the set of all ultrafilters on $\mathbf{A}_P$. Homomorphisms between Lindenbaum algebras are then in correspondence with bounded morphisms between canonical frames. See [BRV01, Chapter 5], [CZ97, Chapter 7] and [Kra99, Chapter 4] for a general introduction to this subject. Duality has already been employed by Ghilardi [Ghi04] to solve unification problems in Heyting algebras. In our work we make substantial use of it to construct a necessary and sufficient condition for $\varphi \in \mathcal{L}_P$ to be projective. Here a central role is played by $\widehat{\varphi}^{\infty} := \bigcap_{n\in\mathbb{N}} \widehat{\Box^n\varphi}$, i.e. the set of all points in $\mathfrak{F}_P$ containing $[\Box^n\varphi]_{\mathbf{L}}$ for all $n \in \mathbb{N}$. Indeed, we prove that φ is projective if and only if there exists a bounded morphism $f : \mathfrak{F}_P \to \mathfrak{F}_P$ such that the image of f is contained in $\widehat{\varphi}^{\infty}$, and all elements of $\widehat{\varphi}^{\infty}$ are fixpoints of f.

*Speaker.

After establishing this equivalence, we apply it to study the projective – or non-projective – character of the extensions of the logics **K4**, **K5**, and

$$\mathbf{K4}_n := \mathbf{K} + (\Diamond^{n+1} p \to \Diamond^{\leq n} p)$$
$$\mathbf{K4}_n\mathbf{B}_k := \mathbf{K4}_n + (p \to \Box^{\leq k} \Diamond^{\leq k} p)$$

where $n, k \geq 1$. We show that all extensions of $\mathbf{K4}_n\mathbf{B}_k$ are projective, thereby reproving a recent result of Kostrzycka [Kos22]. The extensions of **K4** were studied by Kost [Kos18], who proved that the projective extensions of **K4** are exactly the extensions of the logic

$$\mathbf{K4D1} := \mathbf{K4} + \Box(\Box p \to q) \vee \Box(\Box q \to p).$$

Here we show that all locally tabular[1] extensions of **K4D1** are projective, and that all projective extensions of **K4** are also extensions of **K4D1**. This is obviously weaker than Kost's result, but still covers a decent range of logics. With a simple adaptation of our proof, we also show that all projective extensions of $\mathbf{K4}_n$ are extensions of

$$\mathbf{K4}_n\mathbf{D1}_n := \mathbf{K4}_n + \Box(\Box^{\leq n} p \to q) \vee \Box(\Box^{\leq n} q \to p).$$

Most interestingly, we prove that the projective extensions of **K5** are exactly the extensions of **K45**. In particular, this resolves the open question of whether **K5** is projective. It should also be noted that our proofs are fairly lightweight and concise, as opposed to syntactic methods, which often involve all sorts of technical twists. Of course, this is only a first insight of what duality has to offer. Hopefully these results will raise interest in this line of work, and open promising new directions.

References

[BS01] Baader, F., and W. Snyder, 'Unification theory', In: *Handbook of Automated Reasoning,* Elsevier (2001) 439–526.

[BR11] Babenyshev, S., and V. Rybakov, 'Unification in linear temporal logic **LTL**', *Annals of Pure and Applied Logic* **162** (2011) 991–1000.

[BRV01] Blackburn, P., M. de Rijke, and Y. Venema, *Modal Logic,* Cambridge University Press (2001).

[CZ97] Chagrov, A., and M. Zakharyaschev, *Modal Logic,* Oxford University Press (1997).

[Dzi07] Dzik, W., *Unification Types in Logic,* Wydawnicto Uniwersytetu Slaskiego (2007).

[Ghi97] Ghilardi, S., 'Unification through projectivity', *Journal of Logic and Computation* **7** (1997) 733–752.

[Ghi00] Ghilardi, S., 'Best solving modal equations', *Annals of Pure and Applied Logic* **102** (2000) 183–198.

[Ghi04] Ghilardi, S., 'Unification, finite duality and projectivity in varieties of Heyting algebras', *Annals of Pure and Applied Logic* **127**(1-3) (2004) 99–115.

[Kos18] Kost, S., 'Projective unification in transitive modal logics', *Logic Journal of the IGPL* **26** (2018) 548–566.

[Kos22] Kostrzycka, Z., 'Projective unification in weakly transitive and weakly symmetric modal logics', *Journal of Logic and Computation* doi.org/10.1093/logcom/exab081doi.

[Kra99] Kracht, M., *Tools and Techniques in Modal Logic,* Elsevier (1999).

[Sło12] Słomczyńska, K., 'Unification and projectivity in Fregean Varieties', *Logic Journal of the IGPL* **20** (2012) 73–93.

[1] A logic **L** is *locally tabular* if for all finite sets P of variables, there are only finitely many formulas in $\mathcal{L}_P$ modulo $\equiv_{\mathbf{L}}$.